Jacobi-like structures: A line bundle perspective

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Differential operators

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Differential operators. Algebraic

This perspective stems from Vinogradov's Diffiety concept, which is built on the observation that any 'geometry' is the realization of a corresponding 'algebra' [1]. According to [2], at the most elementar level, a differential operator (d.o.) of order p is an additive map

$$\Delta : M \to N, \quad M, N \in \mathrm{Mod}_{\mathrm{R}},$$

that enjoys

$$\delta_{a_0} \cdots \delta_{a_p} \Delta = 0, \quad \delta_a \Delta := [\Delta, a] \, .$$

If the category $\mathrm{Mod}_{\mathrm{R}}$ is refined, i.e., the module structure is enriched, then differential operators should satisfy additional requirements, compatible with the supplementary properties.

e.g. if $\mathrm{Mod}_{\mathrm{R}}$ is replaced by $\mathrm{Mod}_{\mathrm{A}}$, with A an associative, unitary, and commutative algebra over $\mathbb{R}$ then the aditivity should be replaced with $\mathbb{R}$-liniarity.

Here, the ring R is assumed to be unitary and commutative.

Differential operators. Algebraic

The set of d.o. of order p is denoted by $\mathrm{Diff_p(M;N)}$ and it possesses a natural R-ring structure

$$R \times \mathrm{Diff}_p(M;N) \ni (a,\Delta) \mapsto \Delta a \in \mathrm{Diff}_p(M;N), \quad (a\Delta)x := a(\Delta x)$$

These modules exhibit some 'universal' ones,

$$\mathrm{Diff}_p(N), \quad \mathrm{J}^p(M)$$

equipped with 'universal' d.o. of order p

$$\pi_p(N) : \mathrm{Diff}_p(N) \to N, \quad j^p(M) : M \to \mathrm{J}^p(M)$$

that display the isomorphisms

$$\mathrm{Hom}_R(M;\mathrm{Diff}_p(N)) \simeq \mathrm{Diff}_p(M;N) \simeq \mathrm{Hom}_R(\mathrm{J}^p(M);N).$$

Differential operators. Algebraic

The previous 'universal' modules are

$$\mathrm{Diff}_p(N) = \mathrm{Diff}_p(R; N), \quad \mathrm{J}^p(M) = \mathrm{Diff}_p(M; R),$$

while the corresponding isomorphisms read

$$\mathrm{Hom}_R(M; \mathrm{Diff}_p(N)) \ni \Delta \mapsto \pi_p(N) \circ \Delta, \mathrm{Hom}_R(\mathrm{J}^p(M); N) \ni \Delta \mapsto \Delta \circ \mathrm{j}^p(M).$$

The previous isomorphisms can be generalized to multi-additive mappings as

$$\mathrm{Hom}_R(M_1, \cdots, M_k; \mathrm{Diff}_p(N)) \simeq \mathrm{Diff}_p(M_1, \cdots, M_k; N), \tag{1}$$

$$\mathrm{Diff}_p(M_1, \cdots, M_k; N) \simeq \mathrm{Hom}_R(\mathrm{J}^p(M_1), \cdots \mathrm{J}^p(M_k); N). \tag{2}$$

A central role among d.o. is played by the first-order d.o., and moreover, by the derivations. With the previous notations, one denote by $D(N)$, the subset of $\mathrm{Diff}_1(N)$ consisting of additive mappings enjoying the Leibniz rule

$$\Delta(a_1 a_2) = a_1 \Delta a_2 + (\Delta a_1) a_2$$

Differential operators. Algebraic

Contrary to $\mathrm{Diff}_1(N)$, which is an R-module, $D(N)$ is only an Abelian group with respect to punctual addition. At this point it is useful to introduce the definitions associated with arbitrary additive subsets of the module N, $S \subset N$,

$$D(S \subset N) := \{\Delta \in D(N) : \quad \mathrm{Im}\,\Delta \subseteq S\}\,, \tag{3}$$

$$\mathrm{Diff}_p(S \subset N) := \{\Delta \in \mathrm{Diff}_p(N) : \quad \mathrm{Im}\,\Delta \subseteq S\}\,. \tag{4}$$

When S is submodule, $S \leq N$, then

$$D(S \subset N) = D(S), \quad \mathrm{Diff}_p(S \subset N) = \mathrm{Diff}_p(S).$$

The previous definitions further lead to the inclusion

$$D(S \subset N) \subset \mathrm{Diff}_1(S \subset N), \tag{5}$$

that produces

$$D_k(N) \subset D_{k-1}(\mathrm{Diff}_1)(N) \subset D_{k-2}((\mathrm{Diff}_1)^2(N)) \subset \cdots \subset (\mathrm{Diff}_1)^k(N), \tag{6}$$

Differential operators. Algebraic

Here, recursive definitions were used

$$D_k(N) = D\left\{D_{k-1}(N) \subset (\mathrm{Diff}_1)^{k-1}(N)\right\}, D_0(N) = N, D_1(N) = D(N). \quad (7)$$

As the reference module N is arbitrary, then, for any $k \in \mathbb{N}^{\times}$, one can display the sequence of inclusion of functors

$$D_k \subset D_{k-1}(\mathrm{Diff}_1) \subset D_{k-2}(\mathrm{Diff}_1^2) \subset \cdots \subset \mathrm{Diff}_1^k, \quad (8)$$

or, equivalently

$$D_k \xrightarrow{\alpha_k} D_{k-1}(\mathrm{Diff}_1) \xrightarrow{\alpha_{k-1}} D_{k-2}(\mathrm{Diff}_1^2) \xrightarrow{\alpha_{k-2}} \cdots \xrightarrow{\alpha_0} \mathrm{Diff}_1^k. \quad (9)$$

The inclusion of functors α_k, together with the composition of d.o.

$$\mathrm{Diff}_1\left(\mathrm{Diff}_l\right) \xrightarrow{\beta_l} \mathrm{Diff}_{l+1}$$

Differential operators. Algebraic

further yields the functors

$$S^k_{k+l} : D_k\left(\mathrm{Diff}_l\right) \stackrel{\alpha_k(\mathrm{Diff}_l)}{\longrightarrow} D_{k-1}\left(\mathrm{Diff}_1\left(\mathrm{Diff}_l\right)\right) \stackrel{D_{k-1}(\beta_l)}{\longrightarrow} D_{k-1}\left(\mathrm{Diff}_{l+1}\right)$$

which, for each $k \in \mathbb{N}^{\times}$, produce the sequences

$$S_k(N) : 0 \to D_k(N) \stackrel{S^k_k}{\to} D_{k-1}(\mathrm{Diff}_1(N)) \stackrel{S^{k-1}_k}{\to} \cdots \stackrel{S^0_k}{\to} \mathrm{Diff}_k(N) \stackrel{\pi_k(N)}{\to} N$$

The previous sequence $S_k(N)$ is the well-known Spencer complex, found to be exact in its last two factors. This means that $S_1(N)$ is always exact. It has been shown that there are rings for which $S_k(N)$ are exact for an arbitrary module N.

Differential operators

- Algebraic part
- Geometric context

Differential operators. Geometric

Here, the modules in the previous part are naturally associated with vector bundles over a given manifold M. This is because for any vector bundle $E \to M$, the set of its smooth sections $\Gamma(E)$ is a module over the associative, commutative, and unitary algebra $\mathcal{F}(M) := \mathcal{C}^\infty(M;\mathbb{R})$. If $F \to M$ is another vector bundle, one adopts for the $\mathcal{F}(M)$-module of d.o. of order p the notation $\mathrm{Diff}_p(E;F)$. Here, d.o. are $\mathbb{R}$-linear mappings that verify

$$\delta_{a_0} \cdots \delta_{a_p} \Delta = 0, \quad \delta_a \Delta := [\Delta, a] . \tag{10}$$

According to general algebraic description, the module of p-order d.o. coincides with the module of sections

$$\mathrm{Diff}_p(E;F) = \Gamma\left(\mathrm{diff}_k(E;F)\right), \quad \mathrm{diff}_k(E;F) = \left(J^k E\right)^\star \otimes F \tag{11}$$

Obviously, if previously one makes the replacement $F \longrightarrow \mathbb{R}_M$, with $\mathbb{R}_M$ the trivial line bundle over M, $\mathbb{R}_M := \mathbb{R} \times M$, then the identification is manifest

$$J_k E := \mathrm{diff}_k(E; \mathbb{R}_M) = \left(J^k E\right)^\star .$$

To express the Spencer sequences, definition (10) exhibits for each p-order d.o. Δ, the multi-derivation

$$\sigma_\Delta \in \mathrm{Diff}_1(\mathbb{R}_M, \cdots, \mathbb{R}_M; L(E; F))$$

known as the symbol of Δ, which is symmetric in its entries. With the help of this, the Spencer sequence of $\mathbb{R}$-vector spaces reads

$$0 \to \mathrm{Diff}_{p-1}(E; F) \longrightarrow \mathrm{Diff}_p(E; F) \xrightarrow{\sigma} \Gamma\left(\vee^p TM \otimes E^\star \otimes F\right) \to 0, \quad (12)$$

or, equivalently

$$0 \leftarrow \Gamma(J^{p-1}E) \longleftarrow \Gamma(J^p E) \xleftarrow{\gamma} \Gamma\left(\vee^p T^\star M \otimes E\right) \leftarrow 0. \quad (13)$$

Differential operators. Geometric

In the second short sequence, one used the notation γ for the co-symbol

$$\gamma(\mathrm{d}a_1 \vee \cdots \vee \mathrm{d}a_p \otimes e) := (\delta_{a_p} \cdots \delta_{a_1} j^p)e \tag{14}$$

Remark

For $p = 1$ the short sequences (12) and (13) are also exact, but as $\mathbb{R}$-vector spaces.

Among the first-order differential operators, the derivations play a central role in what follows. Let $E \longrightarrow M$ be a vector bundle. An first-order d.o. $\Delta \in \mathrm{Diff}_1(E; E)$ is said to be a derivation if

$$\sigma_\Delta = X_\Delta \otimes \mathrm{id}_{\Gamma(E)}.$$

The set of derivations corresponding to the considered vector bundle $E \longrightarrow M$ is denoted with $\mathcal{D}E$. This is the module of sections in the vector bundle

$$DE \longrightarrow M,$$

Differential operators. Geometric

whose fiber at x, $x \in M$, $(DE)_x$ consisting of $\mathbb{R}$-linear maps $\delta : \Gamma(E) \to E_x$ enjoying of a unique vector $X_\delta \in T_xM$ such that

$$\delta(f\mu) = (X_\delta f)\mu_x + f(x)\delta\mu, \quad f \in \mathcal{F}(M), \quad \mu \in \Gamma(E), \tag{15}$$

i.e.,

$$\mathcal{D}E = \Gamma(DE).$$

The vector bundle $DE \longrightarrow M$ is a Lie algebroid, the well-known Atyiah/gauge algebroid, whose anchor associates to each derivation its symbol

$$(DE)_x \ni \delta \mapsto X_\delta \in T_xM$$

and bracket, the standard commutator

$$[\Delta, \Delta'] := \Delta \circ \Delta' - \Delta' \circ \Delta.$$

It is the vector-bundle version of the ordinary tangent bundle Lie algebroid $TM \longrightarrow M$. It is also functorial [3] with respect to $\mathrm{VB}^{\mathrm{reg}}$.

Differential operators. Geometric

Let $E \longrightarrow M$ and $F \longrightarrow N$ be two vector bundles and $(\phi, \underline{\phi})$ be a regular morphism

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ \Big\downarrow{\scriptstyle \pi_E} & & \Big\downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow{\underline{\phi}} & N \end{array}$$

i.e. it an isomorphism on fibers. In this framework, it is well-defined the pull-back

$$\phi^* : \Gamma(F) \to \Gamma(E),$$

via

$$\Gamma(F) \ni \mu \mapsto (\phi^* \mu)_x := \phi_x^{-1}(\mu_{\underline{\phi}(x)}), \quad x \in M$$

and the map $\phi_* : DE \to DF$ defined on fibers via

$$(\phi_{*,x}\delta)(\mu) := \phi_x(\delta(\phi^* \mu)), \quad \mu \in \Gamma(F).$$

which is proved to be a Lie algebroid map [3,4].

Differential operators. Geometric

Of a very importance in the sequel will be line bundles. These are vector bundles with 1-dimensional fibers. The category of line bundles, Line, possesses products, pull-back [5], and there exist the homogenization and dehomogenization functors to $\mathbb{R}^{\times}$-principal bundles [6]. Also, for line bundles $L \longrightarrow M$, there exists the equality

$$\mathrm{Diff}_1(L) = \mathcal{D}L.$$

For trivial line bundle, $L \longrightarrow \mathbb{R}_M$, the module of sections reduces to

$$\Gamma(\mathbb{R}_M) = \mathcal{F}(M),$$

while the vector bundle of derivations become

$$D\mathbb{R}_M = TM \oplus \mathbb{R}_M.$$

Outline

Lie&Jacobi algebroids: Schouten&Cartan perspectives

Definition

A Lie algebroid is a triple $(A, [\cdot,\cdot], \rho)$ consisting of a vector bundle $A \to M$, an $\mathbb{R}$-Lie algebra structure on $\Gamma(A)$, and a vector bundle map, $\rho : A \to TM$, such that

$$[\alpha, f\beta] = (\rho(\alpha) f)\, \beta + f\, [\alpha, \beta], \quad \alpha, \beta \in \Gamma(A), f \in \mathcal{F}(M).$$

Theorem

Let $A \to M$ be a vector bundle. Then the following ingredients are equivalent:

1. *a Lie algebroid structure, $([\bullet, \bullet], \rho)$, on $A \to M$;*
2. *a Gerstenhaber algebra structure, $[\bullet, \bullet]_A$, on the graded algebra $\mathcal{A}_A^\bullet := \Gamma(\wedge^\bullet A)$;*
3. *a homological degree 1 graded derivation, d_A, of the graded algebra $\tilde{\mathcal{A}}_A^\bullet := \Gamma(\wedge^\bullet A^*)$.*

At the second point of the previous theorem, the Gerstenhaber structure is just the adaptation of Schouten bracket [7] to Lie algebroids, i.e., it is the derivative extension of the bracket $[\bullet,\bullet]$ 'living' on $\Gamma(A)$, to the graded algebra $\mathcal{A}^{\bullet}_{A}$, via the elementary rules

$$[P, Q \wedge R]_A = [P, Q]_A \wedge R + (-)^{(|P|-1)|Q|} Q \wedge [P, R]_A \,, \tag{16}$$

$$[P, Q]_A = -(-)^{(|P|-1)(|Q|-1)} [Q, P]_A \,, \tag{17}$$

for homogeneous elements. The third point in the theorem is the lifting of de Rham calculus to Lie algebroids

$$\langle \mathrm{d}_A \tilde{\omega}, \alpha_0 \wedge \cdots \wedge \alpha_p \rangle = \sum_{j=0}^{p} (-)^j \rho(\alpha_j) \langle \tilde{\omega}, \alpha_0 \wedge \cdots \overset{j}{\wedge} \cdots \wedge \alpha_p \rangle$$
$$+ \sum_{0 \le i < j \le p} (-)^{i+j} \langle \tilde{\omega}, [\alpha_i, \alpha_j] \wedge \alpha_0 \wedge \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots \wedge \alpha_p \rangle. \tag{18}$$

Lie&Jacobi algebroids: Schouten&Cartan perspectives

Example

For the tangent Lie algebroid $(TM, [\bullet,\bullet], \tau_M)$, the Gerstenhaber algebra is that of the multi-vector fields

$$\mathfrak{X}^\bullet(M),$$

equipped with the standard Schouten-Nijenhuis bracket

$$[\![\bullet,\bullet]\!]$$

while the differential one is the well-known de Rham complex

$$\Lambda^\bullet(M),$$

supplied with de Rham differential

$$\mathrm{d}.$$

Lie&Jacobi algebroids: Schouten&Cartan perspectives

$(TM \times \mathbb{R}, [\![\bullet, \bullet]\!], \rho)$ with

$$[\![(X, f), (Y, g)]\!] := ([X, Y], Xg - Yf), \quad \rho(X, f) := X.$$

By means of the isomorphisms

$$\Gamma\left(\wedge^{r+1}(TM \times \mathbb{R})\right) \simeq \mathfrak{X}^{r+1}(M) \times \mathfrak{X}^{r}(M),$$

its Gerstenhaber algebra structure $[\![\bullet, \bullet]\!]$ reads

$$[\![(P, Q), (R, S)]\!] = ([P, R], [P, S] + (-)^{r}[Q, R]).$$

Invoking the isomorphisms

$$\Gamma\left(\wedge^{r+1}(TM \times \mathbb{R})^{*}\right) \simeq \Omega^{r+1}(M) \times \Omega^{r}(M),$$

the homological degree 1 derivation, $\mathbf{d}$, reads

$$\mathbf{d}(\omega, \alpha) := (\mathrm{d}\omega, \mathrm{d}\alpha), \quad (\omega, \alpha) \in \Omega^{k}(M) \times \Omega^{k-1}(M).$$

Definition

Let (A, L) be a pair consisting of a VB $A \to M$ and a LB $L \to M$. A triplet $([\cdot, \cdot], \rho, \nabla)$, where $([\cdot, \cdot], \rho)$ is a Lie algebroid structure on $A \to M$ and ∇ is a flat A-connection on the line bundle $L \to M$, is called a Jacobi/Kirillov [8] algebroid.

Lie&Jacobi algebroids: Schouten&Cartan perspectives

Theorem

Let (A, L) be a pair as before. Denoting by $A_L := A \otimes L^*$ the total space of the VB $A \otimes_M L^*$, the following data are equivalent [9]:

1. *a Jacobi algebroid structure, $([\bullet, \bullet], \rho, \nabla)$, on the pair (A, L);*
2. *a Gerstenhaber-Jacobi algebra structure, $\left([\bullet, \bullet]_{A,L}, X_\bullet^{(A,L)}\right)$, on the graded module $\mathcal{L}^\bullet_{A,L} := \Gamma(\wedge^\bullet A_L \otimes L)[1]$ over the graded algebra $\mathcal{A}^\bullet_{A,L} := \Gamma(\wedge^\bullet A_L)$;*
3. *a homological degree 1 graded derivation, $d_{A,L}$ covering d_A, acting on the graded $\tilde{A}^\bullet_A$-module $\tilde{\mathcal{L}}^\bullet_{A,L} := \Gamma(\wedge^\bullet A^* \otimes L)$.*

The second description in the previous theorem is realized via the relations

$$[\alpha,\beta]_{A,L}=[\alpha,\beta]\,,\quad X_{\alpha}^{(A,L)}f=\rho\left(\alpha\right)f,\quad [\alpha,e]_{A,L}=\nabla_{\alpha}e, \tag{19}$$

with

$$\alpha,\beta\in\Gamma\left(A\right),\quad f\in\mathcal{F}\left(M\right),\quad e\in\Gamma\left(L\right),$$

while the third one is implemented via

$$\left(\mathrm{d}_{A,L}e\right)\left(\alpha\right)=\nabla_{\alpha}e,\quad \alpha\in\Gamma\left(A\right),e\in\Gamma\left(L\right), \tag{20}$$

$$\mathrm{d}_{A,L}\left(\tilde{\omega}\wedge\omega\right)=\left(\mathrm{d}_{A}\tilde{\omega}\right)\wedge\omega+\left(-\right)^{k}\tilde{\omega}\wedge\mathrm{d}_{A,L}\omega,\quad \tilde{\omega}\in\tilde{\mathcal{A}}_{A}^{k},\omega\in\tilde{\mathcal{L}}^{\bullet}{}_{A,L} \tag{21}$$

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two graded left modules over the graded and associative algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. A graded module map/ morphism is a graded $\mathbb{R}$-linear map $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ covering a graded algebra map $\underline{\psi} : \mathcal{A}_1 \to \mathcal{A}_2$ such that

$$\psi(a \cdot m) = \underline{\psi}(a) \cdot \psi(m).$$

Let $(\psi, \underline{\psi})$ be a degree zero graded module map as previously.
A degree k graded $\mathbb{R}$-liniar map $X : \mathcal{A}_1 \to \mathcal{A}_2$ is said to be a degree k graded derivation covering $\underline{\psi}$ if

$$X(ab) = X(a)\underline{\psi}(b) + (-)^{k|a|}\underline{\psi}(a)X(b).$$

By definition, a degree k derivation covering ψ, of symbol X is a degree k graded $\mathbb{R}$-linear map $\Box : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying

$$\Box(a \cdot m) = X(a) \cdot \psi(m) + (-)^{k|a|}\underline{\psi}(a) \cdot \Box m$$

By definition, a Gerstenhaber-Jacobi algebra is a graded module $\mathcal{L}$ over a graded algebra $\mathcal{L}$ which is equipped with a graded Lie algebra structure $[\![\bullet, \bullet]\!]$ and a graded Lie algebra map

$$\mathbb{X} : \mathcal{L} \longrightarrow \mathrm{Der}(\mathcal{A}),$$

that for homogeneous elements verify

$$[\![l, a \cdot l']\!] = X_l(a) \cdot l' + (-)^{|l||a|} a \cdot [\![l, l']\!].$$

When the line bundle is trivial, $L = \mathbb{R}_M$, then the concept of Jacobi algebroid reduces to that of a Lie algebroid with *a* 1*-cocycle* [9], as the structure is completely given by a 1-cocycle

$$\Gamma(A) \ni \alpha \mapsto \nabla_\alpha = \rho(\alpha) + \langle \omega_\nabla, \alpha \rangle \in \mathcal{D}\mathbb{R}_M = \mathfrak{X}^1(M) \oplus \mathcal{F}(M) \tag{22}$$

Moreover, the flatness of A-connection (22) is equivalent

$$\mathrm{d}_A \omega_\nabla = 0. \tag{23}$$

Here, the graded module $\mathcal{L}^\bullet_{A,L}$ become

$$\mathcal{A}^\bullet_{A,L} = \mathcal{A}^\bullet_A := \Gamma(\wedge^\bullet A), \quad \mathcal{L}^\bullet_{A,L} = \mathcal{A}^\bullet_A[1],$$

while the Gerstenhaber-Jacobi algebra structure $\left([\bullet,\bullet]_{A,L}, X^{(A,L)}_\bullet\right)$ reduces to $\left([\bullet,\bullet]^\nabla_A, X^\nabla_\bullet\right)$, where

$$[\alpha, f]^\nabla_A = \nabla_\alpha f, \quad [\alpha,\beta]^\nabla_A = [\alpha,\beta], \quad X^\nabla_\alpha f = \rho(\alpha) f. \tag{24}$$

Finally, the $\tilde{\mathcal{A}}^{\bullet}_{A} := \Gamma\left(\wedge^{\bullet} A^{*}\right)$-module $\tilde{\mathcal{L}}^{\bullet}_{A,L} := \Gamma\left(\wedge^{\bullet} A^{*} \otimes L\right)$ reads

$$\tilde{\mathcal{L}}^{\bullet}_{A,L} = \tilde{\mathcal{A}}^{\bullet}_{A}, \tag{25}$$

while the homological derivation $\mathrm{d}_{A,L}$ (covering d_{A}) becomes d^{∇}_{A}

$$\mathrm{d}^{\nabla}_{A}\omega = \mathrm{d}_{A}\omega + \omega_{\nabla} \wedge \omega. \tag{26}$$

By direct computation it can be proved that homological derivation (26) coincides with de Rham differential associated with graded Lie algebra structure written previously, i.e.,

$$\left\langle \mathrm{d}^{\nabla}_{A}\omega, \alpha_0 \wedge \cdots \wedge \alpha_p \right\rangle = \sum_{j=0}^{p} (-)^{j} \left[\alpha_j, \left\langle \omega, \alpha_0 \wedge \cdots \overset{j}{\wedge} \cdots \alpha_p \right\rangle\right]^{\nabla}_{A}$$

$$+ \sum_{0 \leq i < j \leq p} (-)^{i+j} \left\langle \omega, \left[\alpha_i, \alpha_j\right]^{\nabla}_{A} \wedge \alpha_0 \wedge \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots \wedge \alpha_p \right\rangle. \tag{27}$$

As for any manifold, there exists a natural Lie algebroid; one can naturally associate a Jacobi algebroid (DL, L) with any line bundle $L \to M$Let $L \to M$ be a line bundle. The DL-flat connection is just the tautological representation

$$\nabla : DL \to DL, \quad \nabla_{\square} e := \square e, \quad \square \in \mathcal{D}L, e \in \Gamma(L) . \tag{28}$$

According to Theorem 6, if we adopt the notations

$$[\bullet, \bullet] := [\bullet, \bullet]_{DL} , \quad \sigma := \rho_{DL}, \tag{29}$$

the Jacobi algebroid structure $([\bullet, \bullet], \sigma, \nabla)$ is equivalent to the Gerstenhaber-Jacobi algebra one $\left([\bullet, \bullet] := [\bullet, \bullet]_{DL,L}, X_\bullet := X_\bullet^{(DL,L)}\right)$, on the graded $\mathcal{A}^{\bullet}_{DL,L} := \Gamma(\wedge^\bullet DL_L)$-module $\mathcal{L}^{\bullet}_{DL,L} := \Gamma(\wedge^\bullet DL_L \otimes L)[1]$. At this stage, it is useful to express the previous abstract Gerstenhaber-Jacobi algebra in a more convenient form that allows computations. This assumes both the realization of the algebra $\mathcal{A}^{\bullet}_{DL,L}$ and of the $\mathbb{R}$-vector space $\mathcal{L}^{\bullet}_{DL,L}$.

Lie&Jacobi algebroids: Schouten&Cartan perspectives

In view of this, one uses the implication

$$\left(J^1L\right)^* \otimes L \simeq DL \Rightarrow DL_L := DL \otimes L^* \simeq J_1L := \left(J^1L\right)^* . \tag{30}$$

which, in the light of the general algebraic discussion, further leads to

$$\Gamma\left(\wedge^\bullet J_1L\right) \simeq \mathrm{Diff}\,_1^\bullet\left(L;\mathbb{R}_M\right), \tag{31}$$

and, in addition

$$\mathcal{L}^\bullet_{DL,L} \simeq \mathcal{D}^\bullet L\left[1\right] := \mathrm{Diff}\,_1^\bullet\left(L;L\right)\left[1\right] \Leftrightarrow \mathcal{L}^k_{DL,L} \simeq \mathcal{D}^{k+1}L, \quad k \geq -1. \tag{32}$$

The Gerstenhaber-Jacobi bracket can be written in terms of Gerstenhaber inner multiplication [13]

$$\square \circ \triangle\left(e_1,\ldots,e_{k+l+1}\right) := \sum_{\sigma\in S(l+1,k)} \left(-\right)^\sigma \square\left(\triangle\left(e_{\sigma(1)},\ldots,e_{\sigma(l+1)}\right), e_{\sigma(l+2)},\ldots,e_{\sigma(k+l+1)}\right)$$

as

$$[\![\square,\triangle]\!] := \left(-\right)^{kl}\square\circ\triangle - \triangle\circ\square, \quad \square\in\mathcal{D}^{k+1}L, \triangle\in\mathcal{D}^{l+1}L. \tag{33}$$

To introduce the derivative representation of the module $\mathcal{D}^\bullet L$ on the graded algebra $\mathrm{Diff}\,_1^\bullet(L;\mathbb{R}_M)$, $\mathbb{X}_\bullet$, firstly one defines the *symbol map*

$$\sigma_\square(f)(e_1,\ldots,e_k)\,e := \square(fe,e_1,\ldots,e_k) - f\square(e,e_1,\ldots,e_k)\,, \tag{34}$$

that is a multi-differential, $\sigma_\square(f) \in \mathrm{Diff}\,_1^k(L;\mathbb{R}_M)$. With these specifications at hand, the derivative representation reads

$$\begin{aligned}
&\mathbb{X}_\square\left(\tilde{\Delta}\right)(e_1,\ldots,e_{k+l}) = \\
&(-)^{k(l-1)}\sum_{\sigma\in S(l,k)}(-)^\sigma\,\sigma_\square\left(\tilde{\Delta}\left(e_{\sigma(1)},\ldots,e_{\sigma(l)}\right)\right)\left(e_{\sigma(l+1)},\ldots,e_{\sigma(l+k)}\right) \\
&-\sum_{\sigma\in S(k+1,l-1)}(-)^\sigma\,\tilde{\Delta}\left(\square\left(e_{\sigma(1)},\ldots,e_{\sigma(k+1)}\right),e_{\sigma(k+2)},\ldots,e_{\sigma(k+l)}\right).
\end{aligned}$$

Lie&Jacobi algebroids: Schouten&Cartan perspectives

Finally, de Rham complex associated with the Jacobi algebroid (DL, L) is known as *der-complex* associated with the line bundle $L \to M$ and is commonly denoted by $(\Omega^\bullet_L, \mathrm{d}_L)$. The elements of the graded $\tilde{\mathcal{A}}^\bullet_{DL} := \Gamma\left(\wedge^\bullet (DL)^*\right)$-module $\Omega^\bullet_L := \tilde{\mathcal{L}}^\bullet_{DL,L} := \Gamma\left(\wedge^\bullet (DL)^* \otimes L\right)$ are known as *L-valued Atiyah forms*. Here, the homological degree 1 derivation acts

$$\langle \mathrm{d}_L e, \Box \rangle := \Box e, \quad e \in \Gamma(L), \Box \in \mathcal{D}L, \tag{35}$$

$$\mathrm{d}_L(\tilde{\omega} \wedge \omega) = \mathrm{d}_{DL}\tilde{\omega} \wedge \omega + (-)^k \tilde{\omega} \wedge \mathrm{d}_L \omega, \quad \tilde{\omega} \in \tilde{\mathcal{A}}^k_{DL}, \omega \in \Omega^\bullet_L. \tag{36}$$

The homological derivation enjoys two strong properties: i) it *agrees* with the first-order prolongation and ii) it is *acyclic*.
i) This is expressed by

$$\langle \mathrm{d}_L e, \square \rangle = \langle \square, j^1 e \rangle,$$

where the L-pairing between DL and $J^1 L$ resulting from the isomorphism $DL \simeq (J^1 L)^* \otimes L$ reads

$$\langle \bullet, \bullet \rangle : \mathcal{D}L \times \Gamma(J^1 L) \to \Gamma(L), \quad \langle \square, j^1 e \rangle := \square e. \tag{37}$$

ii) The acyclicity [13] of the homological derivation d_L is done by the existence of a contracting homotopy for $id_{\Omega^\bullet_L}$ with respect to d_L

$$\iota^{(DL,L)}_{\mathrm{id}_{\Gamma(L)}} \mathrm{d}_L + \mathrm{d}_L \iota^{(DL,L)}_{\mathrm{id}_{\Gamma(L)}} = id_{\Omega^\bullet_L}.$$

For trivial line bundle, the Jacobi algebroid $(TM \oplus \mathbb{R}, \mathbb{R}_M)$ structure reduces to

$$[(X,f),(Y,g)] = ([X,Y], Xg - Yf), \quad (X,f),(Y,g) \in \mathfrak{X}^1(M) \oplus \mathcal{F}(M),$$
$$\sigma((X,f)) = X, \quad (X,f) \in \mathfrak{X}^1(M) \oplus \mathcal{F}(M),$$
$$\nabla_{(X,f)} h = Xh + fh, \quad (X,f) \in \mathfrak{X}^1(M) \oplus \mathcal{F}(M), h \in \mathcal{F}(M).$$

Here, to introduce the Gerstenhaber-Jacobi structure over the $\mathrm{Diff}\,^{\bullet}_1(\mathbb{R}_M;\mathbb{R}_M)$ one uses

$$\mathfrak{X}^k(M) \oplus \mathfrak{X}^{k-1}(M) \simeq \mathrm{Diff}\,^k_1(\mathbb{R}_M;\mathbb{R}_M), \quad (P,Q) \longleftrightarrow P + Q \wedge \mathrm{id}, \quad (38)$$

where

$$(P + Q \wedge \mathrm{id})(f_1, \ldots, f_k) := P(f_1, \ldots, f_k) + \sum_{j=1}^{k} (-)^{k-j} Q\left(f_1, \ldots, \widehat{f_j}, \ldots, f_k\right)$$

The Gerstenhaber-Jacobi structure [15] $\left(\llbracket \bullet, \bullet \rrbracket^{(0,1)}, \mathbb{X}_{\bullet}^{(0,1)}\right)$ consists in

$$\llbracket P + Q \wedge \mathrm{id}, R + S \wedge \mathrm{id} \rrbracket^{(0,1)} = [P, R] + p\,(-)^{r}\, P \wedge S - rQ \wedge R$$
$$+ \left([P, S] + (-)^{r}\,[Q, R] + (p - r)\, Q \wedge S\right) \wedge \mathrm{id},$$

and

$$\mathbb{X}_{P+Q\wedge\mathrm{id}}^{(0,1)}\,(R + S \wedge \mathrm{id}) = \llbracket P + Q \wedge \mathrm{id}, R + S \wedge \mathrm{id} \rrbracket^{(0,1)} - Q \wedge R - (Q \wedge S) \wedge \mathrm{id}.$$

Here, the 1-cocycle becomes

$$\omega_{\nabla} = (0,1) \in \Gamma\left((TM \oplus \mathbb{R})^{*}\right) = \Omega^{1}(M) \oplus \mathcal{F}(M),$$

while the *der-complex* $\Omega^{\bullet}_{\mathbb{R}_M}$ is represented as

$$\Omega^{k}\left(M\right) \oplus \Omega^{k-1}\left(M\right) \simeq \Omega^{k}_{\mathbb{R}_M}, \quad \left(\overset{(k)}{\omega}, \overset{(k-1)}{\omega}\right) \longleftrightarrow \overset{(k)}{\omega} + \overset{(k-1)}{\omega} \wedge \mathrm{id}, \tag{39}$$

with

$$\left(\overset{(k)}{\omega} + \overset{(k-1)}{\omega} \wedge \mathrm{id}\right)\left(\left(X_1, f_1\right), \ldots, \left(X_k, f_k\right)\right) := \langle \overset{(k)}{\omega}, X_1 \wedge \cdots \wedge X_k \rangle +$$

$$+ \sum_{j=1}^{k} (-)^{k-j} \langle \overset{(k-1)}{\omega}, X_1 \wedge \cdots \overset{j}{\wedge} \cdots \wedge X_k \rangle f_j. \tag{40}$$

Furthermore, the homological derivation of degree 1 in der-complex, $\mathbf{d}^{(0,1)}$, becomes

$$\mathbf{d}^{(0,1)}\left(\overset{(k)}{\omega} + \overset{(k-1)}{\omega} \wedge \mathrm{id}\right) = \mathrm{d}\overset{(k)}{\omega} + \left(\mathrm{d}\overset{(k-1)}{\omega} + (-)^{k}\overset{(k)}{\omega}\right) \wedge \mathrm{id}. \tag{41}$$

Outline

Integrability results

There are three main results that are commonly invoked when one addresses the integrability problem associated with various Jacobi structures. First, remember that if $\mathcal{E}$ is a everywhere defined smooth distribution on a manifold, $\mathcal{E} \subseteq TM$, then it is said to be *completely integrable* if for any $x \in M$ there exists a submanifold $x \in N \subseteq M$ such that

$$T_y N = \mathcal{E}_y, \quad y \in N.$$

Theorem (Stefan, Sussmann)

Let $\mathcal{E}$ be a distribution as before, $G_{\mathcal{E}}$ be the family of diffeomorphisms generated by $\mathcal{E}$, and $\tilde{\mathcal{E}}$ be the $G_{\mathcal{E}}$-invariant distribution associated with the original one. Then $\tilde{\mathcal{E}}$ is completely integrable [14,15], with the leaf topology identical with that generated by the starting distribution $\tilde{\mathcal{E}}$.

Integrability results

Theorem (Stefan, Sussmann)

With the data previously specified, the following properties are equivalent:

1. *Distribution* $\mathcal{E}$ *is* $G_{\mathcal{E}}$*-invariant;*
2. *Distributions* $\mathcal{E}$ *and* $\tilde{\mathcal{E}}$ *coincides,* $\mathcal{E} = \tilde{\mathcal{E}}$*;*
3. *Distribution* $\mathcal{E}$ *is completely integrable;*
4. *Distribution* $\mathcal{E}$ *is involutive and its rank is constant along the flow lines of its sections.*

Outline

- Differential operators
- Lie&Jacobi algebroids: Schouten&Cartan perspectives
- Integrability results
- Jacobi structures
- Twisted Jacobi structures
- Jacobi structures with background
- References

Jacobi structures

By definition, a Jacobi bundle consists in a line bundle $L \to M$ endowed with a bracket

$$\{\bullet, \bullet\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L),$$

that enjoys the properties:

- It is $\mathbb{R}$-linear and skew-symmetric;
- It verifies the Jacobi identity i.e.

$$\{s_1, \{s_2, s_3\}\} + \text{circular} = 0, \quad s_1, s_2, s_3 \in \Gamma(L) \tag{42}$$

- It is local i.e.

$$\text{supp}\{s_1, s_2\} \subset \text{supp} s_1 \cap \text{supp} s_2, \quad s_1, s_2 \in \Gamma(L) \tag{43}$$

Jacobi structures

In this unified context, a Jacobi bundle consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

that verifies Maurer-Cartan equation

$$[\![J, J]\!] := -2J \circ J = 0. \tag{44}$$

The connection between the bracket and the bi-differential operator J simply reads

$$\{s_1, s_2\} := J\left(s_1, s_2\right), \quad s_1, s_2 \in \Gamma\left(L\right). \tag{45}$$

When the line bundle is trivial, the bi-differential operator $J \in \mathcal{D}^2\mathbb{R}_M$ is expressed in terms a pair (Π, E) as $J = \Pi - E \wedge \mathrm{id}$. With this expression at hand, the Gerstenhaber-Jacobi bracket implies

$$[\Pi, \Pi] + 2\Pi \wedge E = 0, \quad [\Pi, E] = 0. \tag{46}$$

With this expression of the bi-differential operator J, the $\mathbb{R}$-Lie algebra structure over $\mathcal{F}(M)$, $\{\bullet, \bullet\}$, reduces to the well-known Jacobi bracket

$$\{f, g\} = i_\Pi \left(\mathrm{d}f \wedge \mathrm{d}g\right) + i_E \left(f\mathrm{d}g - g\mathrm{d}f\right), \quad f, g \in \mathcal{F}(M).$$

Jacobi structures

By means of the vector bundle morphism

$$\hat{J} : J^1L \wedge J^1L \to L, \quad \langle \hat{J}, j^1\lambda \wedge j^1\rho \rangle := J(\lambda, \rho), \tag{47}$$

the Jacobi bundle $(L \to M, J)$ is said to be transitive if

$$\mathrm{Im}\left(\sigma \circ \hat{J}^\sharp\right) = TM.$$

Example

A hyperplane distribution $\mathcal{K}$ on M is said to be a contact structure on M if its'curvature'

$$\omega_{\mathcal{K}} : \mathcal{K} \times \mathcal{K} \to TM/\mathcal{K}, \quad \langle \omega_{\mathcal{K}}, X \wedge Y \rangle := [X, Y] \mod \mathcal{K}$$

is non-degenerate. It defines a unique Jacobi bundle $(TM/\mathcal{K} \to M, J_{\mathcal{K}})$ which is transitive.

Example

An lcs structure on a given line bundle $L \to M$ is a pair (∇, Ω) consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle and a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ which is closed with respect to the homological degree 1 derivation d_∇ associated with the Jacobi algebroid structure $([\bullet, \bullet], \nabla)$ on the pair (TM, L),

$$d_\nabla \Omega = 0.$$

It defines a unique transitive Jacobi bundle $(L \to M, J)$ with

$$J(\lambda, \mu) := \langle \Omega, \Omega^\sharp(\mathrm{d}_\nabla \mu) \wedge \Omega^\sharp(\mathrm{d}_\nabla \lambda) \rangle.$$

Jacobi structures

Theorem

Let $(L \to M, J)$ be a transitive Jacobi bundle. Then the following alternative holds.

- *If the base manifold is even-dimensional then the considered Jacobi bundle is equivalent to a locally conformal symplectic structure on the same line bundle.*
- *If the base manifold is odd-dimensional then the considered Jacobi bundle is equivalent to a contact structure displaying the same line bundle.*

Theorem

The characteristic distribution of a Jacobi bundle is completely integrable with the characteristic leaves either locally conformal symplectic manifolds or contact ones.

Outline

Twisted Jacobi structures

A twisted Jacobi bundle consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

which 'nilpotency' (44) is 'twisted' via the closed Atiyah 3-form

$$\Phi \in \Omega_L^3, \quad \mathrm{d}_L \Phi = 0 \Leftrightarrow \Phi = \mathrm{d}_L \Omega, \tag{48}$$

i.e.

$$[\![J, J]\!] = 2 \left(\wedge^3 \hat{J}^\sharp \right)^* \mathrm{d}_L \Omega. \tag{49}$$

Also here, the twisted Jacobi bundle $(L \to M, J, \Omega)$ is said to be transitive if

$$\mathrm{Im} \left(\sigma \circ \hat{J}^\sharp \right) = TM.$$

Twisted Jacobi structures

The structure is invariant under the transformation

$$\Omega \longrightarrow \Omega + \mathrm{d}_L \tilde{\Omega},$$

which means that in the trivial line bundle case [16], one can choose

$$\Omega = \omega \in \Omega^2(M),$$

and

$$\begin{aligned} \tfrac{1}{2}\left[\Pi, \Pi\right] + E \wedge \Pi &= \wedge^3 \Pi^\sharp \mathrm{d}\omega + \wedge^2 \Pi^\sharp \omega \wedge E, \\ [E, \Pi] &= -\left(\wedge^2 \Pi^\sharp i_E \mathrm{d}\omega + \Pi^\sharp i_E \omega \wedge E\right). \end{aligned}$$

Twisted Jacobi structures

Example

A hyperplane distribution $\mathcal{K}$ together with a 2-form $\psi \in \Gamma\left(\wedge^2\mathcal{K}^* \otimes L\right)$, $L := TM/\mathcal{K}$ is said to be a twisted contact structure on M if

$$\omega_{\mathcal{K}} + \psi \in \Gamma\left(\wedge^2\mathcal{K}^* \otimes L\right)$$

is non-degenerate. It defines a unique twisted Jacobi bundle $(L \to M, J_{\mathcal{K},\psi}, \Omega_{\mathcal{K},\psi})$ which is transitive.

Twisted Jacobi structures

Example

A twisted lcs structure on a given line bundle $L \to M$ is pair $((\nabla, \Omega), \omega)$ consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle, a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ and an L-valued 2-form $\omega \in \Omega^2(M; L)$ which verify the compatibility condition

$$d_\nabla \Omega = d_\nabla \omega.$$

It defines a unique transitive twisted Jacobi bundle $(L \to M, J, \mathrm{d}_D \sigma^* \omega)$ with

$$J(\lambda, \mu) := \langle \Omega, \Omega^\sharp(\mathrm{d}_\nabla \mu) \wedge \Omega^\sharp(\mathrm{d}_\nabla \lambda) \rangle.$$

Twisted Jacobi structures

Theorem

Let $(L \to M, J, \Omega)$ *be a twisted Jacobi bundle, which is transitive. Then the following alternative holds.*

- *If the base manifold is even-dimensional then the considered Jacobi bundle is equivalent to a twisted locally conformal symplectic structure on the same line bundle.*
- *If the base manifold is odd-dimensional then the considered Jacobi bundle is equivalent to a twisted contact structure displaying the same line bundle.*

Theorem

The characteristic distribution of a twisted Jacobi bundle is completely integrable [17] with the characteristic leaves either twisted locally conformal symplectic manifolds or twisted contact ones.

Outline

Jacobi structures with background

Finally, a Jacobi bundle with background consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

whose 'nilpotency' (44) is 'broken' via an Atiyah 3-form

$$\Phi \in \Omega_L^3, \tag{50}$$

i.e.

$$[\![J, J]\!] = 2 \left(\wedge^3 \hat{J}^\sharp \right)^* \Phi. \tag{51}$$

Jacobi structures with background

For trivial line bundle situation, the Atyiah 3-form

$$\Phi = \phi + \omega \wedge \mathrm{id}, \quad \phi \in \Omega^3(M), \omega \in \Omega^2(M). \tag{52}$$

By inserting the previous data in structure equation, one derives [18]

$$\frac{1}{2}[\Pi, \Pi] + E \wedge \Pi = \wedge^3 \Pi^\sharp \phi + \wedge^2 \Pi^\sharp \omega \wedge E, \tag{53}$$

$$[E, \Pi] = -\left(\wedge^2 \Pi^\sharp i_E \phi + \Pi^\sharp i_E \omega \wedge E\right) \tag{54}$$

Jacobi structures with background

The bi-differential operator J exhibits the correspondence

$$\Gamma(L) \ni e \mapsto \triangle_e \in \mathcal{D}^1 L, \quad \triangle_e := -[\![J, e]\!] := \hat{J}^\sharp(j^1 e),$$

which displays Hamiltonian derivations. Theirs symbols

$$X_e := \sigma(\triangle_e)$$

are the well-known Hamiltonian vector fields. If we use the notation

$$\{e_1, e_2\} := J(e_1, e_2) = \triangle_{e_1} e_2, \quad e_1, e_2 \in \Gamma(L), \tag{55}$$

then

$$[\triangle_{e_1}, \triangle_{e_2}] = \triangle_{\{e_1,e_2\}} + \hat{J}^\sharp \left(\iota^{(DL,L)}_{\triangle_{e_2}} \iota^{(DL,L)}_{\triangle_{e_1}} \Phi \right),$$
$$[X_{e_1}, X_{e_2}] = X_{\{e_1,e_2\}} + \sigma \circ \hat{J}^\sharp \left(\iota^{(DL,L)}_{\triangle_{e_2}} \iota^{(DL,L)}_{\triangle_{e_1}} \Phi \right).$$

Jacobi structures with background

The category of Jacobi bundles with background is completed by morphisms of Jacobi bundles with background, i.e., Jacobi maps. Let $(L_i \to M_i, J_i, \Psi_i)$, $i = 1, 2$ be two Jacobi bundles with background. A regular vector bundle morphism (i.e. fiber-wise isomorphism) $\varphi : L_1 \to L_2$ covering $\underline{\varphi} \in \mathcal{C}^\infty(M_1, M_2)$ is said to be a Jacobi map iff

$$\varphi^*\{\lambda, \mu\}_2 = \{\varphi^*\lambda, \varphi^*\mu\}_1, \quad \lambda, \mu \in \Gamma(L_2). \tag{56}$$

Previously, by φ^* we denoted the pull-back associated with the given regular vector bundle morphism

$$\varphi^* : \Gamma(L_2) \to \Gamma(L_1), \quad (\varphi^*\mu)(x) := (\varphi_x)^{-1} \mu\left(\underline{\varphi}(x)\right), \quad x \in M_1.$$

Jacobi structures with background

Let $(L_i \to M_i)$, $i = 1, 2$ be two line bundles and $\left(\varphi, \underline{\varphi}\right)$ be a regular morphism between them. We define the pull-back to $\varphi^* : \Gamma(J^1 L_2) \to \Gamma(J^1 L_1)$ as

$$\varphi^* \left(j^1 e_2\right) := j^1 \left(\varphi^* e_2\right), \quad e_2 \in \Gamma(L_2)$$

and extend it via $\mathbb{R}$-linearity and semi-linearity to the whole module $\Gamma(J^1 L_2)$ as

$$\varphi^* \left(f j^1 e_2\right) := (\underline{\varphi}^* f) j^1 \left(\varphi^* e_2\right), \quad f \in \mathcal{F}(M_2), e_2 \in \Gamma(L_2).$$

Definition

The derivations $\triangle_1 \in \mathcal{D}L_1$ and $\triangle_2 \in \mathcal{D}L_2$ are said to be $\left(\varphi, \underline{\varphi}\right)$-related if

$$\varphi_*(\triangle_1) = \triangle_2$$

Jacobi structures with background

Theorem

Le t $(L_i \to M_i, J_i, \Psi_i)$, $i = 1, 2$ *be two Jacobi bundles with background. A regular vector bundle morphism (i.e. fiber-wise isomorphism)* $\varphi : L_1 \to L_2$ *covering* $\underline{\varphi} \in \mathcal{C}^\infty(M_1, M_2)$ *is a Jacobi map iff*

$$\hat{J}_2^\sharp = D\varphi \circ \hat{J}_1^\sharp \circ \varphi^*.$$

Theorem

Let $(L_i \to M_i, J_i, \Psi_i)$, $i = 1, 2$ *be two Jacobi bundles with background. A regular vector bundle morphism (i.e. fiber-wise isomorphism)* $\varphi : L_1 \to L_2$ *covering* $\underline{\varphi} \in \mathcal{C}^\infty(M_1, M_2)$ *is a Jacobi map iff for any section* $e_2 \in \Gamma(L_2)$, *the Hamiltonian derivations* $\triangle_{\varphi^* e_2}$ *and* $\triangle_{e_2}$ *are* $(\varphi, \underline{\varphi})$*-related*

$$\varphi_*(\triangle_{\varphi^* e_2}) = \triangle_{e_2}.$$

Jacobi structures with background

A Jacobi bundle with background $(L \to M, J, \Phi)$ is said to be transitive if

$$\mathrm{Im}\left(\sigma \circ \hat{J}^{\sharp}\right) = TM.$$

Example

An lcs structure with background on a given line bundle $L \to M$ is pair $((\nabla, \Omega), (\phi, \omega))$ consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle, a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ an L-valued 3-form $\phi \in \Omega^3(M; L)$ and an L-valued 2-form which verify the compatibility condition

$$d_{\nabla}\Omega = d_{\nabla}\omega + \phi.$$

It defines a unique transitive Jacobi bundle with background$(L \to M, J, \mathrm{d}_D\sigma^*\omega + \sigma^*\phi)$ with

$$J(\lambda, \mu) := \langle \Omega, \Omega^{\sharp}(\mathrm{d}_{\nabla}\mu) \wedge \Omega^{\sharp}(\mathrm{d}_{\nabla}\lambda) \rangle.$$

Jacobi structures with background

Starting with the Spencer short exact sequence

$$\Gamma\left(T^*M\otimes L\right) := \Omega^1(M;L) \overset{\gamma}{\rightarrowtail} \Gamma(J^1L) \overset{\pi_{1,0}}{\twoheadrightarrow} \Gamma(L)$$

(which particulary splits as short exact sequence of $\mathbb{R}$-vector spaces by the first-order prolongation $j^1 : \Gamma(L) \to \Gamma(J^1L)$) we consistently define the bi-symbol of J, $\tilde{J} \in \Gamma\left(\wedge^2(T^*M\otimes L)^*\otimes L\right)$, via

$$\langle\tilde{J}, \eta\wedge\theta\rangle := \langle\hat{J}, \gamma(\eta)\wedge\gamma(\theta)\rangle, \quad \eta,\theta\in\Omega^1(M;L).$$

The bi-symbol $\tilde{J}$, combined with the pairing $L^*\otimes L = \mathbb{R}_M$ display the vector bundle morphism

$$\tilde{J}^\sharp : T^*M\otimes L \to TM, \quad \langle\theta_2, \tilde{J}^\sharp(\theta_1\otimes e_1)\rangle e_2 := \langle\tilde{J}, (\theta_1\otimes e_1)\wedge(\theta_2\otimes e_2)\rangle \quad (57)$$

which enjoys

$$\tilde{J}^\sharp = \sigma\circ\hat{J}^\sharp\circ\gamma. \quad (58)$$

Jacobi structures with background

Theorem

Let $(L \to M, J, \Phi)$ be a Jacobi bundle with background. Then, the following conditions are equivalent:

- *$\hat{J}$ is non-degenerate, and*
- *the base manifold M is odd-dimensional and the Jacobi structure is transitive.*

Theorem

Let $(L \to M, J, \Phi)$ be a Jacobi bundle with background. Then, the following conditions are equivalent:

- *the bi-symbol $\tilde{J}$ is non-degenerate, and*
- *the base manifold M is even-dimensional and the Jacobi structure is transitive.*

Jacobi structures with background

Theorem

Let $(L \to M, J, \Psi)$ be a Jacobi bundle with background, which is transitive. Then the following alternative holds.

- *If the base manifold is even-dimensional then the considered Jacobi bundle is equivalent to a locally conformal symplectic structure with background on the same line bundle.*
- *If the base manifold is odd-dimensional then the considered Jacobi bundle is equivalent to a twisted contact structure displaying the same line bundle.*

Theorem

The characteristic distribution of a Jacobi bundle with background is completely integrable with the characteristic leaves either locally conformal symplectic manifolds with background or twisted contact ones.

Outline

- Differential operators
- Lie&Jacobi algebroids: Schouten&Cartan perspectives
- Integrability results
- Jacobi structures
- Twisted Jacobi structures
- Jacobi structures with background
- References

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