Lie group representations and standard subspaces of Hilbert spaces

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Topics

1. The 2nd quantization and Lie group representations
2. Application to standard subspaces of a complex Hilbert space
3. Holomorphic extension of 1-parameter groups on loc. conv. spaces
4. Analyticity domains for semisimple Lie groups

The second quantization: Fock space

1. First quantization: symplectic manifold $(M, \omega) \rightsquigarrow$ complex Hilbert space $\mathcal{H}$
2. Second quantization: functor $\mathcal{H} \mapsto \mathfrak{F}(\mathcal{H})$ on the category of complex Hilbert spaces

- $\mathcal{H}$ complex Hilbert space
- for every $n \geq 1$,
  - $\mathfrak{S}_n \times \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$, $\sigma.(v_1 \otimes \cdots \otimes v_n) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$
  - $S^n\mathcal{H} := (\mathcal{H}^{\otimes n})^{\mathfrak{S}_n} := \{v \in \mathcal{H}^{\otimes n} \mid (\forall \sigma \in \mathfrak{S}_n) \quad \sigma.v = v\}$
- $S^0\mathcal{H} := \mathbb{C}$
- $\mathfrak{F}_+(\mathcal{H}) := \bigoplus\limits_{n \geq 0} S^n\mathcal{H}$ (bosonic Fock space)
- $\mathrm{Exp} \colon \mathcal{H} \to \mathfrak{F}_+(\mathcal{H})$, $\mathrm{Exp}\,(v) := \bigoplus_{n \geq 0} \frac{1}{\sqrt{n!}} v^{\otimes n}$
- $\Omega := \exp_{\mathcal{H}} 0 \in S^0\mathcal{H} \subseteq F(\mathcal{H})$ ("vacuum vector")

Example: $\mathcal{H} = \mathbb{C}$
$\rightsquigarrow \mathcal{F}_+(\mathcal{H}) = L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \mathrm{d}x) \ni \mathrm{e}^{vx - v^2/2} = \mathrm{Exp}\,(v)$ for $v \in \mathbb{C}$, $\Omega = 1$

The second quantization: standard subspaces

- $\mathrm{Exp}\colon \mathcal{H} \to \mathfrak{F}_+(\mathcal{H})$ homeo. onto a lin. indep., *total* subset of $\mathfrak{F}_+(\mathcal{H})$
- $\mathbf{U}\colon \mathcal{H} \to \mathcal{U}(\mathfrak{F}_+(\mathcal{H}))$, $a \mapsto \mathbf{U}_a$
  - $\mathbf{U}_a(\mathrm{Exp}\,(v)) = \mathrm{e}^{-(\|a\|^2/2+\langle a,v\rangle)}\mathrm{Exp}\,(v+a)$
  - $\mathbf{U}_a\mathbf{U}_b = \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\langle a,b\rangle}\mathbf{U}_{a+b}$ (canonical commutation relations)

The Grassmann manifold of a Banach space $\mathcal{X}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is

$$\mathrm{Gr}(\mathcal{X}) := \{\mathtt{V} \subseteq \mathcal{X} \mid (\exists P = P^2 \in \mathcal{L}(\mathcal{X})) \quad P(\mathcal{X}) = \mathtt{V}\}$$

Thm. (H. Araki, 1963: quantum theory of a free Bose field)

- $\mathcal{R}\colon \mathrm{Gr}(\mathcal{H}_{\mathbb{R}}) \to \mathrm{Gr}(\mathcal{L}(\mathfrak{F}_+(\mathcal{H})))$, $\quad \mathcal{R}(\mathtt{V}) := \mathbf{U}(\mathtt{V})''$
- $\mathcal{R}(\mathtt{V})' = \mathcal{R}(\mathtt{V}')$, where $\mathtt{V}' := \mathtt{V}^{\perp_{\mathrm{Im}\langle\cdot,\cdot\rangle}}$
- $\Omega \begin{cases}\text{cyclic} \\ \text{separating}\end{cases}$ for $\mathcal{R}(\mathtt{V}) \iff \begin{cases}\overline{\mathtt{V}+\mathrm{i}\mathtt{V}} = \mathcal{H} \\ \mathtt{V}\cap \mathrm{i}\mathtt{V} = \{0\}\end{cases} \overset{\text{not}}{\iff} \mathtt{V} \in \mathrm{Gr}_{\mathrm{std}}(\mathcal{H}_{\mathbb{R}})$

This motivates the search for standard subspaces of $\mathcal{H}$

- M. Rieffel – A. van Daele (1975-1977) rediscovered the standard subspaces in their bounded operator approach to modular theory.

Modular data of standard subspaces

- If $\mathtt{V} \in \mathrm{Gr}_{\mathrm{std}}(\mathcal{H}_{\mathbb{R}})$ then $S_{\mathtt{V}} \colon \mathtt{V} + \mathrm{i}\mathtt{V} \to \mathtt{V} + \mathrm{i}\mathtt{V}$, $v + \mathrm{i}w \mapsto v - \mathrm{i}w$, is a closed, anti-linear, densely-defined operator with the polar decomp.
  - $S_{\mathtt{V}} = J_{\mathtt{V}} \Delta_{\mathtt{V}}^{1/2}$, $\Delta_V \geq 0$, $J_{\mathtt{V}} = J_{\mathtt{V}}^{-1}$ anti-linear isometry
  - $(\forall t \in \mathbb{R}) \quad J_{\mathtt{V}} \Delta_{\mathtt{V}}^{\mathrm{i}t} = \Delta_{\mathtt{V}}^{\mathrm{i}t} J_{\mathtt{V}}$
- The correspondence

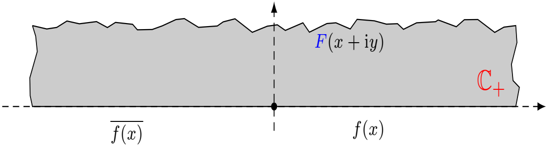
$$\mathtt{V} \mapsto (J_{\mathtt{V}}, (\Delta_{\mathtt{V}}^{\frac{t}{2\pi \mathrm{i}}})_{t \in \mathbb{R}})$$

is 1-to-1 from $\mathrm{Gr}_{\mathrm{std}}(\mathcal{H}_{\mathbb{R}})$ onto the pairs $(J, (U_t)_{t \in \mathbb{R}})$ with $(U_t)_{t \in \mathbb{R}}$ cont. unitary group, $J = J^{-1}$ anti-lin. isometry, $JU_t = U_t J$ for all $t \in \mathbb{R}$. The inverse correspondence is

$$(J, (U_t = \Delta^{\frac{t}{2\pi \mathrm{i}}})_{t \in \mathbb{R}}) \mapsto \mathrm{Ker}(\Delta^{1/2} - J) =: \mathtt{V}$$

(R. Longo (2008))

Example of standard subspace with $\mathtt{V} \dotplus \mathrm{i}\mathtt{V} \subsetneqq \mathcal{H}$



The Hardy space $H^2(\mathbb{C}_+) := \left\{F \in \mathcal{O}(\mathbb{C}_+) : \sup_{y>0} \|F(\cdot + \mathrm{i}y)\|_{L^2(\mathbb{R})} < \infty\right\}$

has the isometric boundary-value operator $H^2(\mathbb{C}_+) \hookrightarrow L^2(\mathbb{R};\mathbb{C})$, $F \mapsto F|_{\mathbb{R}}$.

- $\mathcal{H} := L^2(\mathbb{R}_+)$, $S = S^{-1}\colon H^2(\mathbb{C}_+)|_{\mathbb{R}_+} \to H^2(\mathbb{C}_+)|_{\mathbb{R}_+}$, $F(z) \mapsto \overline{F(-\bar{z})}$
- $\mathtt{V} := \{f = F|_{\mathbb{R}_+} \in H^2(\mathbb{C}_+)|_{\mathbb{R}_+} : SF = F, \text{i.e.}, F(-x) = \overline{f(x)}\}$

$\rightsquigarrow \mathtt{V} \subset \mathcal{H}$ is standard with $\mathtt{V} \dotplus \mathrm{i}\mathtt{V} = H^2(\mathbb{C}_+)|_{\mathbb{R}_+} \subsetneqq L^2(\mathbb{R}_+) = \mathcal{H}$.

We can view $H^2(\mathbb{C}_+) \hookrightarrow L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$ as the graph of the densely-defined, closed, linear operator $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_-)$ defined by analytic extension from $\mathbb{R}_+$ to $\mathbb{C}_+$.

E.g., the Gaussian function $F_a(z) := \mathrm{e}^{-az^2}$ satisfies $F_a \in \mathtt{V}$ for all $a > 0$.

The idea of holomorphic extension of 1-parameter groups

- $\mathcal{Y}$ Hausdorff locally convex space/$\mathbb{C}$
- $(U_t)_{t\in\mathbb{R}}$ 1-parameter subgroup of $\mathrm{GL}(\mathcal{Y})$, i.e.,
  - $(\forall t, s \in \mathbb{R}) \quad U_{t+s} = U_t U_s$
  - $(\forall v \in \mathcal{Y}) \quad U^v \in \mathcal{C}(\mathbb{R}, \mathcal{Y})$, where $U^v \colon \mathbb{R} \to \mathcal{Y}$, $U^v(t) := U_t v$
- the subspace of entire vectors w.r.t. $(U_t)_{t\in\mathbb{R}}$ is

$$\mathcal{Y}^{\mathcal{O}} := \{v \in \mathcal{Y} \mid (\exists F_v \in \mathcal{O}(\mathbb{C}, \mathcal{Y}))(\forall t \in \mathbb{R}) \quad F_v(t) = U_t v\}$$

$\rightsquigarrow (\forall z \in \mathbb{C}) \quad U_z \colon \mathcal{Y}^{\mathcal{O}} \to \mathcal{Y}^{\mathcal{O}},\ U_z v := F_v(z)$

Example: For the Fréchet space $\mathcal{Y} := \mathcal{C}(\mathbb{R}, \mathbb{C})$ with the 1-parameter group

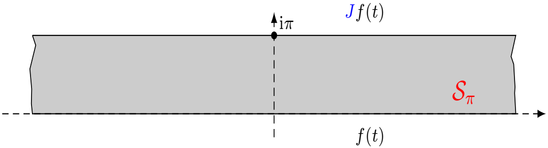
$$(U_t v)(x) := v(x+t) \quad \text{for} \quad t, x \in \mathbb{R}, v \in \mathcal{C}(\mathbb{R}, \mathbb{C})$$

we have

$$\mathcal{Y}^{\mathcal{O}} = \{v \in \mathcal{Y} : (\exists \widetilde{v} \in \mathcal{O}(\mathbb{C})) \quad \widetilde{v}|_{\mathbb{R}} = v\} \simeq \boxed{\mathcal{O}(\mathbb{C}) \hookrightarrow \mathcal{C}(\mathbb{R}, \mathbb{C})} = \mathcal{Y}.$$

Kubo-Martin-Schwinger (KMS) boundary condition

- $\mathcal{S}_\pi := \mathbb{R} + \mathrm{i}(0,\pi) \subset \mathbb{R} + \mathrm{i}[0,\pi] =: \overline{\mathcal{S}}_\pi \subset \mathbb{C}$
- $\mathcal{Y}$ Hausdorff locally convex space/$\mathbb{C}$
- $(U_t)_{t\in\mathbb{R}}$ 1-parameter subgroup of $\mathrm{GL}(\mathcal{Y})$
- $J\colon \mathcal{Y} \to \mathcal{Y}$ anti-linear continuous
- compatibility condition: $(\forall t \in \mathbb{R}) \quad JU_t = U_tJ$



- $v \in \mathcal{Y}$ satisfies the KMS condition ($v \in \mathcal{Y}_{\mathrm{KMS}}$) if there is $f \in \mathcal{C}(\overline{\mathcal{S}}_\pi, \mathcal{Y})$,
  - $f|_{\mathcal{S}_\pi}\colon \mathcal{S}_\pi \to \mathcal{Y}$ weakly holomorphic,
  - $(\forall t \in \mathbb{R}) \quad f(t) = U_t v$, $\boxed{f(t+\mathrm{i}\pi) = Jf(t)}$ $(= JU_t v)$.

Other occurrences of the KMS condition:

1. quantum statistical mechanics, in connection with the Gibbs states;
2. Tomita-Takesaki theory, in connection with the modular automorphism group associated to a faithful normal semifinite weight of a W^*-algebra.

Standard subspaces in terms of KMS conditions

- Let $\mathtt{V} \in \mathrm{Gr}_{\mathrm{std}}(\mathcal{H}_{\mathbb{R}})$ with $S_{\mathtt{V}} \colon \mathtt{V} + \mathrm{i}\mathtt{V} \to \mathtt{V} + \mathrm{i}\mathtt{V}$, $v + \mathrm{i}w \mapsto v - \mathrm{i}w$, and
  - $S_{\mathtt{V}} = J_{\mathtt{V}} \Delta_{\mathtt{V}}^{1/2}$ polar decomp., $J_{\mathtt{V}} = J_{\mathtt{V}}^{-1}$ anti-linear isometry
  - $(\forall t \in \mathbb{R}) \quad J_{\mathtt{V}} \Delta_{\mathtt{V}}^{\mathrm{i}t} = \Delta_{\mathtt{V}}^{\mathrm{i}t} J_{\mathtt{V}}$

Then $\mathtt{V} = \mathcal{H}_{\mathrm{KMS}}$ w.r.t. $J_{\mathtt{V}}$ and the 1-parameter group $(\Delta_{\mathtt{V}}^{\frac{t}{2\pi\mathrm{i}}})_{t \in \mathbb{R}}$
(K.-H. Neeb, B. Ørsted, G. Ólafsson (2021))

Smooth vectors for Lie group representations

- G finite-dim. real Lie group with its Lie algebra $\mathfrak{g}$, and $\exp_G\colon \mathfrak{g} \to G$
- $U\colon G \to \mathcal{U}(\mathcal{H})$, $g \mapsto U_g$, unitary repres., i.e., a group morphism and
  - $(\forall v \in \mathcal{H}) \quad U^v \in \mathcal{C}(G, \mathcal{H})$, where $U^v\colon G \to \mathcal{H}$, $U^v(g) := U_g v$
- $\mathcal{H}^\infty := \{v \in \mathcal{H} \mid U^v \in \mathcal{C}^\infty(G, \mathcal{H})\}$ the smooth vectors (dense in $\mathcal{H}$)
- $\partial U\colon \mathfrak{g} \to \mathrm{End}\,(\mathcal{H}^\infty)$, $\partial U(x)v := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} U_{\exp_G tx} v$

Fix a basis $x_1, \dots, x_m$ in $\mathfrak{g}$ and denote $\mathcal{I}_k := \{1, \dots, m\}^k$, $\mathcal{I} := \bigsqcup\limits_{k \in \mathbb{Z}_+} \mathcal{I}_k$.

For any $k \in \mathbb{Z}_+$ and $I = (i_1, \dots, i_k) \in \mathcal{I}_k$ we define

$$p_I\colon \mathcal{H}^\infty \to \mathbb{R}_+, \quad p_I(\xi) := \|\partial U(x_{i_1}) \cdots \partial U(x_{i_k})\xi\|.$$

A Fréchet topology on $\mathcal{H}^\infty$ is defined by the seminorms $\mathcal{P}_0 := \{p_I : I \in \mathcal{I}\}$.

Exponential growth on the space of smooth vectors

Let $\mathcal{P}$ be the set of finite sums of seminorms in $\mathcal{P}_0$.

Exponential growth on smooth vectors

- $(\forall x \in \mathfrak{g})(\forall p \in \mathcal{P})(\exists q \in \mathcal{P})(\exists A, M > 0)(\forall t \in \mathbb{R})(\forall v \in \mathcal{H}^\infty)$
  $$\boxed{p(U_{\exp_G tx} v) \le M \mathrm{e}^{A|t|} q(v)}$$
- $\lim\limits_{g \to \mathbf{1}} U_g = \mathrm{id}$ uniformly on every bounded subset of $\mathcal{H}^\infty$

• $\mathcal{H}^\omega := \{v \in \mathcal{H} \mid U^v \in \mathcal{C}^\omega(G, \mathcal{H})\}$ the analytic vectors ($\subseteq \mathcal{H}^\infty$)
Fact (J. Frahm, K.-H. Neeb, G. Ólafsson (2023)): The space $\mathcal{H}^\omega$ is a countable inductive limit of Fréchet spaces and carries a continuous representation $U^\omega \colon G \to \mathcal{L}(\mathcal{H}^\omega)$. Moreover $\mathcal{H}^\omega$ is dense in $\mathcal{H}^\infty$.

Exponential growth estimates as above do *not* hold in general on $\mathcal{H}^\omega$.

Holomorphic extension of 1-parameter groups on LCS

- $\mathcal{Y}$ Hausdorff LCS (locally convex space) $/\mathbb{C}$
- $(U_t)_{t\in\mathbb{R}}$ 1-parameter group in $\mathrm{GL}(\mathcal{Y})$
- If $\Gamma \subseteq \mathbb{C}$, $\mathcal{O}_\partial(\Gamma, \mathcal{Y}) := \{F \in \mathcal{C}(\Gamma, \mathcal{Y}) \mid F \text{ weakly holomorphic on } \mathrm{int}\,\Gamma\}$
- If $\varepsilon_1 \le 0 \le \varepsilon_2$, $\overline{\mathcal{S}}_{\varepsilon_1,\varepsilon_2} := \{z \in \mathbb{C} \mid \varepsilon_1 \le \mathrm{Im}\, z \le \varepsilon_2\}$, $\mathcal{S}_{\varepsilon_1,\varepsilon_2} := \mathrm{int}\,\overline{\mathcal{S}}_{\varepsilon_1,\varepsilon_2}$,

$$\mathcal{Y}_{\varepsilon_1,\varepsilon_2} := \{y \in \mathcal{Y} \mid (\exists F_y \in \mathcal{O}_\partial(\overline{\mathcal{S}}_{\varepsilon_1,\varepsilon_2}, \mathcal{Y}))(\forall t \in \mathbb{R})\ F_y(t) = U_t y\}$$

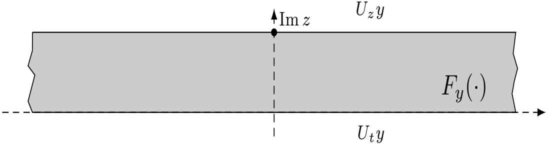
$$\mathcal{Y}^{\mathcal{O}} := \bigcap_{\varepsilon_1\le 0\le \varepsilon_2} \mathcal{Y}_{\varepsilon_1,\varepsilon_2} = \{y \in \mathcal{Y} \mid (\exists F_y \in \mathcal{O}(\mathbb{C}, \mathcal{Y}))(\forall t \in \mathbb{R})\ F_y(t) = U_t y\}$$

Def. For every $z \in \mathbb{C}$ we define $\boxed{U_z \colon \mathcal{D}(U_z) \to \mathcal{Y}}$ by

$$\mathcal{D}(U_z) := \begin{cases} \mathcal{Y}_{0,\mathrm{Im}\, z} & \text{if } \mathrm{Im}\, z > 0, \\ \mathcal{Y} & \text{if } \mathrm{Im}\, z = 0, \\ \mathcal{Y}_{\mathrm{Im}\, z, 0} & \text{if } \mathrm{Im}\, z < 0 \end{cases} \quad \text{and } U_z y := F_y(z) \text{ if } y \in \mathcal{D}(U_z).$$

Holomorphic extension of 1-parameter groups on LCS (2)

For $\operatorname{Im} z > 0$ we have



1. $(\forall z \in \mathbb{C})$ U_z injective, $\operatorname{Ran} U_z = \mathcal{D}(U_{-z})$, and $U_z^{-1} = U_{-z}$.
2. If $z, w \in \mathbb{C}$ and $(\operatorname{Im} z)(\operatorname{Im} w) \geq 0$ then $U_w U_z = U_{w+z}$.

Exponential growth of 1-parameter groups on LCS

- $\mathcal{Y}$ sequentially complete Hausdorff LCS $/\mathbb{C}$
- $U = (U_t)_{t\in\mathbb{R}}$ 1-parameter group in $\mathrm{GL}(\mathcal{Y})$ has exponential growth if

$$(\forall p \in \mathcal{P})(\exists q \in \mathcal{P})(\exists A, M > 0)(\forall t \in \mathbb{R})(\forall y \in \mathcal{Y}) \quad p(U_t y) \le M \mathrm{e}^{A|t|} q(y)$$

- $\mathcal{A} := \{\varphi \in \mathcal{C}(\mathbb{R}, \mathbb{C}) \mid (\exists M, a > 0)(\forall t \in \mathbb{R}) \quad |\varphi(t)| \le M \mathrm{e}^{-t^2/a}\} \subset L^1(\mathbb{R})$

Ex.:: The function $\gamma_{a,z}(t) := \frac{1}{\sqrt{\pi a}} \mathrm{e}^{-(t-z)^2/a}$ for $t \in \mathbb{R}$, satisfies $\gamma_{a,z} \in \mathcal{A}$ if $a > 0, z \in \mathbb{C}$.

Approximation Procedure

$U\colon \mathcal{A} \to \mathcal{L}(\mathcal{Y})$, $U(\varphi) := \int_{\mathbb{R}} \varphi(t) U_t \mathrm{d}t$ (weakly convergent)

- U is a well-defined algebra homomorphism
- $\lim\limits_{a\to 0} U(\gamma_{a,0}) = \mathrm{id}$ uniformly on every compact subset of $\mathcal{Y}$
- $(\forall a > 0, y \in \mathcal{Y}) \quad F_{a,y}\colon \mathbb{C} \to \mathcal{Y}$, $F_{a,y}(z) := U(\gamma_{a,z})y$, satisfies $F_{a,y} \in \mathcal{O}(\mathbb{C}, \mathcal{Y})$ and $F_{a,y}(t) = U_t U(\gamma_{a,0})y$ if $t \in \mathbb{R}$

The holomorphic extension is densely defined

- $\mathcal{Y}$ sequentially complete Hausdorff LCS $/\mathbb{C}$
- $U = (U_t)_{t\in\mathbb{R}}$ 1-parameter group in $\mathrm{GL}(\mathcal{Y})$ with exponential growth

Corollary 1

The space of entire vectors

$$\mathcal{Y}^{\mathcal{O}} = \{y \in \mathcal{Y} \mid (\exists F_y \in \mathcal{O}(\mathbb{C}, \mathcal{Y}))(\forall t \in \mathbb{R}) \quad F_y(t) = U_t y\}$$

is sequentially dense in $\mathcal{Y}$.

Corollary 2

For every $z \in \mathbb{C}$ the linear operator $U_z \colon \mathcal{D}(U_z) \to \mathcal{Y}$ is sequentially densely defined in $\mathcal{Y}$.

Proof: $\mathcal{Y}^{\mathcal{O}} \subseteq \mathcal{D}(U_z)$

Duality theory for holomorphic extensions

- $\mathcal{Y}$ Hausdorff LCS $/\mathbb{C}$
- $\mathcal{Y}^\sharp$ the space of continuous anti-linear functionals on $\mathcal{Y}$
- $\mathcal{Y}^\sharp_c$ the space $\mathcal{Y}^\sharp$ with the topology of uniform convergence on the compact subsets of $\mathcal{Y}$

Duality Theorem

- $\mathcal{Y}$ Fréchet space
- $U = (U_t)_{t\in\mathbb{R}}$ 1-parameter group in $\mathrm{GL}(\mathcal{Y})$ with exponential growth
- $(\forall t \in \mathbb{R}) \quad V_t := U_t^\sharp \in \mathrm{GL}(\mathcal{Y}^\sharp_c)$

$\Longrightarrow \ (\forall z \in \mathbb{C}) \quad V_{\overline{z}} = U_z^\sharp$

It suffices to assume that the canonical map $\mathcal{Y} \to (\mathcal{Y}^\sharp_c)^\sharp$ is surjective.

L. Zsidó (1977) obtained a result of this type for dual pairs of Banach spaces satisfying suitable topological conditions.

Closedness properties of holomorphic extensions

Closed Operator Theorem

- $\mathcal{Y}$ Fréchet space
- $U = (U_t)_{t\in\mathbb{R}}$ 1-parameter group in $\mathrm{GL}(\mathcal{Y})$ with exponential growth

$\Longrightarrow$ $(\forall z \in \mathbb{C})$ $\quad U_z \colon \mathcal{D}(U_z) \to \mathcal{Y}$ closed operator

Again, it suffices to assume that the canonical map $\mathcal{Y} \to (\mathcal{Y}_c^\sharp)^\sharp$ is surjective.

I. Ciorănescu and L. Zsidó (1976) obtained such a result for dual pairs Banach spaces satisfying suitable topological conditions.

Corollary

If in addition $J \colon \mathcal{Y} \to \mathcal{Y}$ is a continuous anti-linear operator, then for every $z \in \mathbb{C}$ we have

1. $\mathrm{Ker}(V_z - J^\sharp) = ((U_{\bar z} - J)\mathcal{D}(U_{\bar z}))^\perp$
   w.r.t. $\langle\cdot,\cdot\rangle_{\mathbb{R}} := \mathrm{Re}\langle\cdot,\cdot\rangle \colon \mathcal{Y} \times \mathcal{Y}^\sharp \to \mathbb{R}$
2. $\mathrm{Ker}(V_z - J^\sharp)$ *is a closed $\mathbb{R}$-linear subspace of* $\mathcal{Y}^\sharp$

Application to standard subspaces

- G finite-dim. real Lie group with its Lie algebra $\mathfrak{g}$, and $\exp_G\colon \mathfrak{g} \to G$
- $U\colon G \to \mathcal{U}(\mathcal{H})$, $g \mapsto U_g$, unitary repres.
- $\mathcal{H}^\infty := \{v \in \mathcal{H} \mid U^v \in \mathcal{C}^\infty(G, \mathcal{H})\}$ smooth vectors (Fréchet space $\hookrightarrow \mathcal{H}$)
- $\mathcal{H}^{-\infty} := (\mathcal{H}^\infty)^\sharp$ distribution vectors (cont. anti-lin. functionals on $\mathcal{H}^\infty$)
- $\partial U\colon \mathfrak{g} \to \mathrm{End}\,(\mathcal{H}^\infty)$, $\partial U(x)v := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} U_{\exp_G tx} v$
- $U^{\pm\infty}\colon G \to \mathcal{L}(\mathcal{H}^{\pm\infty})$
- $h \in \mathfrak{g}$; $(\forall t \in \mathbb{R})$ $\quad U_{h,t} := U_{\exp_G th} \in \mathcal{U}(\mathcal{H})$
- $J\colon \mathcal{H} \to \mathcal{H}$ anti-lin. isometry satisfying $J^2 = \mathrm{id}$ and
  - $JU_{h,t} = U_{h,t}J$ for $t \in \mathbb{R}$
  - $J\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$, which leads to $J^{\pm\infty}\colon \mathcal{H}^{\pm\infty} \to \mathcal{H}^{\pm\infty}$

Application to standard subspaces (2)

- $\Delta := e^{2\pi i \partial U(h)} > 0$ satisfies $\boxed{\Delta^{\frac{t}{2\pi i}} = U_{h,t}}$ for $t \in \mathbb{R}$;

$\rightsquigarrow \mathtt{V} := \{v \in \mathcal{D}(\Delta^{1/2}) \mid \Delta^{1/2} v = Jv\}$ standard subspace in $\mathcal{H}$

- $\mathcal{H}^{-\infty}_{\mathrm{KMS}} := (\mathcal{H}^{-\infty})_{\mathrm{KMS}}$ w.r.t. $(U^{-\infty}_{h,t})_{r\in\mathbb{R}}$ and $J^{-\infty}$

Hence $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ is the space of all distribution vectors $\xi \in \mathcal{H}^{-\infty} = (\mathcal{H}^{\infty})^{\sharp}$ for which there is $f \in \mathcal{C}(\overline{\mathcal{S}}_{\pi}, \mathcal{H}^{-\infty})$, satisfying

- $f|_{\mathcal{S}_\pi} : \mathcal{S}_\pi \to \mathcal{H}^{-\infty}$ weakly holomorphic,
- $(\forall t \in \mathbb{R}) \quad f(t) = U^{-\infty}_{h,t}\xi,\ f(t + i\pi) = J^{-\infty} f(t))$

where $\mathcal{S}_\pi = \mathbb{R} + i(0,\pi) \subset \overline{\mathcal{S}}_\pi \subset \mathbb{C}$

Approximation Theorem

1. $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ weak*-closed in $\mathcal{H}^{-\infty}$
2. $\mathcal{H}^{-\infty}_{\mathrm{KMS}} \cap \mathcal{H} = \mathtt{V}$
3. $\mathtt{V}$ weak*-dense in $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$

The affine group of the real line $G = \mathbb{R} \rtimes \mathbb{R}_+^\times$

- $X := \mathbb{R}$, $\Xi := \mathbb{R} \simeq \mathbb{R}^*$
- $G := \{x \mapsto b + ax \mid a \in \mathbb{R}_+^\times, b \in \mathbb{R}\} \simeq \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}_+^\times, b \in \mathbb{R} \right\}$
- $\mathfrak{g} = \mathbb{R}e + \mathbb{R}h$, $h = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $[h, e] = e$

- $U \colon G \to \mathcal{U}(L^2(X))$, $(U_{(b,a)} f)(x) := a^{-1/2} f(a^{-1}(x - b))$
- $J \colon L^2(X) \to L^2(X)$, $Jf := \overline{f}$
  satisfies $JU_{(b,a)} = U_{(b,a)} J$ for all $(b, a) \in G$
- $\partial U(h) = \frac{\mathrm{d}}{\mathrm{d}b}\Big|_{b=0} U_{(b,1)} = -\frac{\mathrm{d}}{\mathrm{d}x}$

The Fourier transform

$$\mathcal{F} \colon L^2(X) \to L^2(\Xi),\ (\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi x} f(x) \mathrm{d}x$$

moves U from $L^2(X)$ to $L^2(\Xi)$...

The affine group of the real line $G = \mathbb{R} \rtimes \mathbb{R}^\times_+$ (2)

Recall: $X = \mathbb{R}$, $\Xi = \mathbb{R}$

- $\widehat{U} := \mathcal{F} U \mathcal{F}^{-1} \colon G \to \mathcal{U}(L^2(\Xi))$, $(\widehat{U}_{(b,a)}\varphi)(\xi) = a^{-1/2}\mathrm{e}^{\mathrm{i}b\xi}\varphi(a\xi)$
- $\partial\widehat{U}(h) = \frac{\mathrm{d}}{\mathrm{d}b}\Big|_{b=0}\widehat{U}_{(b,1)} = \mathrm{mult}(\mathrm{i}\xi)$ (multiplication oper. in $L^2(\Xi)$)

$\implies \widehat{\Delta}^{1/2} = \mathrm{e}^{\pi \mathrm{id}\widehat{U}(h)} = \mathrm{mult}(\mathrm{e}^{-\pi\xi})$

- $\widehat{J} := \mathcal{F} J \mathcal{F}^{-1} \colon L^2(\Xi) \to L^2(\Xi)$, $(\widehat{J}\varphi)(\xi) := \overline{\varphi(-\xi)}$

- $\widehat{\mathtt{V}} := \{\varphi \in L^2(\Xi) \mid \widehat{\Delta}^{1/2}\varphi = \widehat{J}\varphi\}$

$\Rightarrow$ $\boxed{\widehat{\mathtt{V}} = \{\varphi \in L^2(\Xi) \mid \mathrm{e}^{-\pi\xi}\varphi(\xi) = \overline{\varphi(-\xi)} \text{ a.e.}\} \text{ standard subsp. in } L^2(\Xi)}$

i.e., $\widehat{\mathtt{V}} \cap \mathrm{i}\widehat{\mathtt{V}} = \{0\}$ and $\widehat{\mathtt{V}} + \mathrm{i}\widehat{\mathtt{V}}$ dense in $L^2(\Xi)$

- $\widehat{S} \colon \widehat{\mathtt{V}} + \mathrm{i}\widehat{\mathtt{V}} \to \widehat{\mathtt{V}} + \mathrm{i}\widehat{\mathtt{V}}$, $(\widehat{S}\varphi)(\xi) := \mathrm{e}^{\pi\xi}\overline{\varphi(-\xi)}$

$\implies \widehat{S}$ anti-lin., closed, densely defined, $\widehat{S}^2 = \mathrm{id}$, $\widehat{V} = \mathrm{Ker}(\widehat{S} - \mathrm{id})$

- $\mathtt{V} := \mathcal{F}^{-1}\widehat{\mathtt{V}}$ standard subsp. in $L^2(X)$

Beyond exponential growth: analytic vectors for 1-parameter groups

- $U = (U_t = \mathrm{e}^{\mathrm{i}tA})_{t\in\mathbb{R}}$ continuous 1-parameter unitary group on $\mathcal{H}$
- For every $t > 0$ we define the Hilbert space

$$\mathcal{H}^\omega_t(U) := \mathcal{D}(\underbrace{\mathrm{e}^{tA}}_{U_{-\mathrm{i}t}}) \cap \mathcal{D}(\underbrace{\mathrm{e}^{-tA}}_{U_{\mathrm{i}t}}) = \mathcal{D}(\mathrm{e}^{t|A|}) \qquad (\hookrightarrow \mathcal{H})$$

- The space of *analytic vectors* for U is the countable inductive limit of Hilbert spaces $\mathcal{H}^\omega(U) := \bigcup\limits_{t>0} \mathcal{H}^\omega_t(U) = \varinjlim\limits_{n\in\mathbb{N}} \mathcal{H}^\omega_n(U)$.
- The space of *hyperfunction vectors* for U is the anti-dual space

$$\mathcal{H}^{-\omega} := (\mathcal{H}^\omega)^\sharp$$

and it has the Fréchet topology of a projective limit of Hilbert spaces.

The affine group of the real line $G = \mathbb{R} \rtimes \mathbb{R}_+^\times$ (3)

Recall:

- $G := \{x \mapsto b + ax \mid a \in \mathbb{R}_+^\times, b \in \mathbb{R}\} \simeq \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}_+^\times, b \in \mathbb{R} \right\}$
- $\mathfrak{g} = \mathbb{R}e + \mathbb{R}h$, $[h, e] = e$
- $\Xi = \mathbb{R}$, $\Xi_\pm := \pm(0, \infty)$, $L^2(\Xi) = L^2(\Xi_+) \oplus L^2(\Xi_-)$

$\implies \widehat{U} = \widehat{U}^+ \oplus \widehat{U}_-$

$\widehat{U}^\pm \colon G \to \mathcal{U}(L^2(\Xi_\pm))$, $(\widehat{U}^\pm_{(b,a)}\varphi)(\xi) = a^{-1/2}\mathrm{e}^{\mathrm{i}b\xi}\varphi(a\xi)$ irreducible repres.

We denote $\mathcal{H} := L^2(\Xi_+)$

- $\widehat{U}^+_h := (\widehat{U}^+_{\exp_G th} = \widehat{U}^+_{(0,\mathrm{e}^t)})_{t\in\mathbb{R}}$, $\quad \widehat{U}^+_e := (\widehat{U}^+_{\exp_G be} = \widehat{U}^+_{(b,1)})_{b\in\mathbb{R}}$
- $[h, e] = e \implies \widehat{U}^+_h$ restricts to a cont. 1-param. group on $\mathcal{H}^\omega(\widehat{U}^+_e)$

Let $v \in \mathcal{H}^\omega(\widehat{U}^+) = \mathcal{H}^\omega(\widehat{U}^+_h) \cap \mathcal{H}^\omega(\widehat{U}^+_e)$. If there exists $t > \pi/2$ satisfying

- $v \in \mathcal{D}(\mathrm{e}^{\pm\mathrm{i}t\mathrm{d}U(h)})$
- $(\forall r \in (0, t)) \quad \mathrm{e}^{\mathrm{i}r\mathrm{d}U(h)}v \in \mathcal{H}^\omega(\widehat{U}^+)$

then $v = 0$.

$\rightsquigarrow$ The 1-parameter group $\widehat{U}^+_h$ has no nonzero entire vectors in the Fréchet space $\mathcal{H}^{-\omega}(\widehat{U}^+_e)$, hence it does *not* have polynomial growth.

Applications to semisimple Lie groups

Let $(U, \mathcal{H})$ be a unitary representation of a Lie group G with $G \hookrightarrow G_{\mathbb{C}}$.
Question: If $0 \neq v \in \mathcal{H}^\omega$, how large is the domain in $G_{\mathbb{C}}$ of the analytic extension of the orbit map $U^v \colon G \to \mathcal{H}$, $U^v(g) := U_g v$?

Theorem

Let G be a semisimple Lie group with trivial fixed point space $\mathcal{H}^G = \{0\}$. If $x \in \mathfrak{g}$ is hyperbolic, i.e., $\operatorname{ad} x$ is diagonalizable, and $0 \neq v \in \mathcal{H}^\omega$ is such that $v \in \mathcal{D}(\mathrm{e}^{\mathrm{i}t\partial U(x)})$ and $\mathrm{e}^{\mathrm{i}t\partial U(x)} v \in \mathcal{H}^\omega$ for $|t| \leq 1$, then all eigenvalues of $\operatorname{ad} x$ belong to the interval $[-\pi/2, \pi/2]$.

Applications to semisimple Lie groups (2)

For a unitary representation $(U, \mathcal{H})$ of a Lie group G, $0 \neq v \in \mathcal{H}^\omega$ and $x \in \mathfrak{g}$, we define the *analyticity radius*

$$r_v(x) := \sup\{r > 0\colon v \in \mathcal{D}(\mathrm{e}^{\pm \mathrm{i} r \partial U(x)}), (\forall |t| < r)\ \mathrm{e}^{\pm \mathrm{i} t \partial U(x)} v \in \mathcal{H}^\omega\}.$$

We also consider $r_U(x) := \sup\{r_v(x)\colon 0 \neq v \in \mathcal{H}^\omega\} \in (0, \infty]$.

- (a) The function $r_U\colon \mathfrak{g} \to (0, \infty]$ is $\mathrm{Ad}(G)$-invariant and $r_U(\lambda x) = |\lambda|^{-1} r_U(x)$ for $\lambda \neq 0$.
- (b) For $h, x \in \mathfrak{g}$ with $[h, x] = x \neq 0$, we have $r_U(x) = \infty$.
- (c) Suppose that G is semisimple and $\mathcal{H}^G = \{0\}$. Then we have:
  - (i) $r_U(x) = \infty$ if x is nilpotent or if x is elliptic and U is irreducible.
  - (ii) $r_U(h) \leq \frac{\pi}{2\rho(\mathrm{ad}\, h)}$ if $0 \neq h$ is hyperbolic. Equality holds if G is linear and U is irreducible.
  - (iii) If $x = x_h + x_e$ is semisimple, where x_h is hyperbolic, x_e elliptic and $[x_h, x_e] = 0$, G is linear, and the representation U is irreducible, then $r_U(x) \geq r_U(x_h)$.