HOMOGENEITY AND FORMALISMS OF MECHANICS

Janusz Grabowski
(Polish Academy of Sciences)

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Euler's Homogeneous Function Theorem

Let us start with an easy student exercise in the first course of Calculus, known as Euler's Homogeneous Function Theorem:

Proposition

Any C^1-differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ is homogeneous of degree 1, i.e., a function satisfying

$$f(t\cdot x)=t\cdot f(x) \quad \textit{for all} \quad t>0\,, \tag{1}$$

if and only if f is linear.

It is clear that we can replace $\mathbb{R}^n$ with any n-dimensional real vector space E, and the condition (1) with $\nabla_E(f)=f$, where ∇_E is the Euler vector field on E; in homogeneous coordinates,

$$\nabla_E=\sum x^i\partial_{x^i}\,.$$

Hence, the dual space E^* can be defined as the space of smooth 1-homogeneous functions, so the linear structure on E is determined by the multiplication by reals, $h_t(v)=tv$.

Vector bundles

Our observation can be naturally extended to vector bundles.

Traditionally, a vector bundle is defined as a locally trivial fibration $\tau : E \to M$ with an atlas of local trivializations $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ such that the transition maps are linear in fibers,

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n ,$$

where $A(x, \cdot) \in \mathrm{GL}(n, \mathbb{R})$.

The latter property can also be expressed in terms of the multiplication by reals $h_t(x, y) = (x, ty)$:

$$A \circ h_t = h_t \circ A \quad \text{for all} \quad t \in \mathbb{R}.$$

It follows that the multiplication by reals is well-defined globally, $h_t : E \to E$, and completely determines the vector bundle structure. The projection $\tau : E \to M$ is simply h_0.

Consequences for vector bundles

Since working with vector bundles we can reduce ourselves to the multiplication by reals $h_t(v) = t \cdot v$, one proves the following.

Corollary (Grabowski-Rotkiewicz 2009)

- *A smooth map $\Phi : E_1 \to E_2$ between the total spaces of two vector bundles $\pi_i : E_i \to M_i$, $i = 1, 2$, is a morphism of vector bundles if and only if it intertwines the multiplications by reals:*

  $$\Phi(t \cdot v) = t \cdot \Phi(v)\,.$$

  In this case, the map $\varphi = \Phi_{|M_1}$ is a smooth map between the base manifolds covered by Φ.
- *Vector subbundles of a vector bundle $\tau : E \to M$ are smooth submanifolds $E_0 \subset E$ which are invariant with respect to the multiplication by reals, $h_t(E_0) \subset E_0$. In this case, E_0 is itself a vector bundle over $M_0 = \tau(E_0) = M \cap E_0$ and the multiplication by reals inherited from E.*

Graded bundles

A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ modelled on a graded vector space

$$\mathbb{R}^{\mathbf{d}} = \mathbb{R}^{d_1}[1] \times \cdots \times \mathbb{R}^{d_k}[k].$$

We call k the degree and $\mathbf{d} = (d_1, \ldots, d_k)$ the rank of F, and view linear coordinates (y^a_w) in $\mathbb{R}^{d_w}[w]$ as being homogeneous of degree (weight) w with respect to the canonical dilation $h^{\mathbf{d}}$,

$$h^{\mathbf{d}}_t(y_1, \ldots, y_k) = (t \cdot y_1, \ldots, t^k \cdot y_k)\,, \quad y_w \in \mathbb{R}^{d_w}\,, \quad t \in \mathbb{R}\,,$$

i.e., $y^a_w \circ h^{\mathbf{d}}_t = t^w \cdot y^a_w$.

More precisely, F is a fiber bundle with the typical fiber $\mathbb{R}^{\mathbf{d}}$ and with an atlas of local trivializations $\tau^{-1}(U) \simeq U \times \mathbb{R}^{\mathbf{d}}$ such that the transition maps respect the dilation $h_t(x, y) = (x, h^{\mathbf{d}}_t(y))$,

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^{\mathbf{d}}\,,$$

$$A \circ h_t = h_t \circ A \quad \text{for all} \quad t \in \mathbb{R}.$$

Like for vector bundles, the dilation $h_t : F \to F$ is globally defined and $\tau = h_0$.

Weight vector fields and homogeneity

Note that our graded bundles are not graded vector bundles, since the transition maps are generally not linear on the graded vector space $\mathbb{R}^{\mathbf{d}}$: for $h_t(y,z)=(ty,t^2z)$ on $\mathbb{R}^2$ we have $\varphi\circ h_t=h_t\circ\varphi$, where $\varphi(y,z)=(y,z+y^2)$, but φ is not linear.

Like in the case of a vector space, we have local homogeneous coordinates (x^A,y^a_w) on F, where the coordinates x^A on M are of degree 0, and the coordinates y^a_w in fibers are of degree w. The dilation h_t is completely determined by the weight vector field

$$\nabla_F=\sum_{w=1}^{k}\sum_{a=1}^{d_w} w\cdot y^a_w\partial_{y^a_w}\,.$$

We call a smooth function $f:F\to\mathbb{R}$ homogeneous of degree (weight) $\alpha\in\mathbb{R}$ if $f\circ h_t=t^\alpha\cdot f$ for $t>0$ ($\nabla_F(f)=\alpha\cdot f$).

A morphism of graded bundles is a smooth map respecting homogeneity degrees of functions, i.e., relating the corresponding weight vector fields $\to$ the category GrB.

Graded bundles are polynomial

Generally, we call a tensor field K on F homogeneous of degree (weight) $\alpha \in \mathbb{R}$ if

$$\pounds_{\nabla_F} K = \alpha \cdot K.$$

Proposition (Grabowski-Rotkiewicz 2012)

If $f : F \to \mathbb{R}$ is a homogeneous function of degree α, then $\alpha \in \mathbb{N}$ and f is locally a polynomial in homogeneous fiber coordinates y^a_w, with coefficients being smooth functions in the base coordinates (x^A). Consequently, morphisms of graded bundles are polynomial in local homogeneous coordinates of degree > 0. In particular, the transition functions $A(x, y)$ are polynomial in variables (y^a_w), i.e., any graded bundle is a polynomial bundle.

Note that vector bundles are just graded bundles of degree 1. Another trivial example is a split graded bundle, i.e. a graded vector bundle

$$F = E^1[1] \oplus_M \cdots \oplus_M E^k[k] ,$$

where E^i are vector bundles over M.

A canonical example

Example. Consider the second-order tangent bundle $\mathsf{T}^2 M$, i.e., the bundle of second jets of smooth paths $(\mathbb{R}, 0) \to M$. Writing the Taylor expansion of paths in local coordinates (x^A) on M:

$$x^A(t) = x^A(0) + \dot{x}^A(0)t + \ddot{x}^A(0)\frac{t^2}{2} + o(t^2)\,,$$

we get local coordinates $(x^A, \dot{x}^B, \ddot{x}^C)$ on $\mathsf{T}^2 M$, which transform

$$\begin{aligned} x'^A &= x'^A(x)\,, \\ \dot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x)\,\dot{x}^B\,, \\ \ddot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x)\,\ddot{x}^B + \frac{\partial^2 x'^A}{\partial x^B \partial x^C}(x)\,\dot{x}^B\dot{x}^C\,. \end{aligned}$$

Hence, associating with $(x^A, \dot{x}^B, \ddot{x}^C)$ the weights $0, 1, 2$, we get a graded bundle structure of degree 2 on $\mathsf{T}^2 M$.

Due to the quadratic terms above, this is not a vector bundle!

All this generalizes to higher tangent bundles $\mathsf{T}^k M$ which are canonically graded bundles od degree k.

Transition functions for graded bundles

In fact, the form of transition maps (changes of local coordinates) for a general graded bundle is quite similar,

$$\begin{aligned} x'^A &= x'^A(x), \\ y'^a_w &= y^b_w \cdot T_b{}^a(x) + \sum_{\substack{1<n \\ w_1+\cdots+w_n=w}} \frac{1}{n!} y'^{b_1}_{w_1} \cdots y'^{b_n}_{w_n} \cdot T_{b_n \cdots b_1}{}^a(x), \end{aligned}$$

where $T_b{}^a$ are invertible and $T_{b_n \cdots b_1}{}^a$ are symmetric in indices b.

Note that the transition functions of coordinates of degree r involve only coordinates of degree $\leq r$, that defines a reduced graded bundle F_r of degree r (we simply 'forget' coordinates of degrees $> r$). Moreover, they are linear in coordinates of degree r modulo a shift by a polynomial in variables of degrees $< r$,

$$y'^a_r = y^b_r \cdot T_b{}^a(x) + \sum_{\substack{1<n \\ w_1+\cdots+w_n=r}} \frac{1}{n!} y'^{b_1}_{w_1} \cdots y'^{b_n}_{w_n} \cdot T_{b_n \cdots b_1}{}^a(x),$$

so the fibrations $F_r \to F_{r-1}$ are affine.

Graded bundles - the tower of affine fibrations

In this way, for any graded bundle F of degree k we get a tower of affine fibrations

$$F = F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

Note that the bundles in the tower are only affine, so there is no canonical embedding of F_{r-1} into F_r nor F.

Example

In the case of the canonical graded bundle $F = \mathsf{T}^k M$, we get exactly the tower of natural projections of jet bundles

$$\mathsf{T}^k M \xrightarrow{\tau^k} \mathsf{T}^{k-1} M \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} \mathsf{T}^2 M \xrightarrow{\tau^2} \mathsf{T} M \xrightarrow{\tau^1} F_0 = M\,.$$

For a split graded bundle we have

$$E^1[1] \oplus_M \cdots \oplus_M E^k[k] \to E^1[1] \oplus_M \cdots \oplus_M E^{k-1}[k-1] \to \cdots$$
$$\cdots \to E^1[1] \oplus_M E^2[2] \to E^1[1] \to M.$$

Homogeneity structures

The multiplication by reals in a vector bundle and, more generally, the dilations

$$h : \mathbb{R} \times F \to F, \quad h(t,p) = h_t(p),$$

which for a graded bundle $\tau : F \to M$ with local homogeneous coordinates (x^A, y^a_w) read

$$h_t(x^A, y^a_w) = (x^A, t^w y^a_w)\,,$$

represent smooth actions of the monoid (not a group!) $(\mathbb{R}, \cdot)$ of multiplicative reals: $h_1 = \mathrm{id}_F, \quad h_t \circ h_s = h_{ts}.$

Such actions of $(\mathbb{R}, \cdot)$ we will call homogeneity structures.

This is because h defines the concept of homogeneity on F: $f : F \to \mathbb{R}$ is homogeneous of degree $\alpha \in \mathbb{R}$ if $f \circ h_t = t^\alpha \cdot f$ for $t > 0$. We can also use the associated weight vector field,

$$\nabla(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} h_t(p),$$

and say that a tensor field K on F is homogeneous of degree α if $\pounds_\nabla(K) = \alpha \cdot K$.

Graded bundle = homogeneity structure

We know that with every graded bundle there is canonically associated a homogeneity structure.

The fundamental result in the theory of graded bundles says that graded bundles and homogeneity structures are, in fact, fully equivalent concepts.

Theorem (Grabowski-Rotkiewicz 2012)

Associating canonically the homogeneity structure with a graded bundle yields an isomorphism of categories.

More precisely, for any homogeneity structure $h : \mathbb{R} \times F \to F$ *on a manifold* F*, the subset* $M = h_0(F)$ *of* F *is a smooth submanifold and there is a non-negative integer* $k \in \mathbb{N}$ *such that* $h_0 : F \to M$ *is canonically a graded bundle of degree* k *whose homogeneity structure coincides with* h*. In other words, there is an atlas on* F *consisting of local homogeneous functions.*

Double vector bundles

An important concept of a double vector bundle, introduced by Pradines in 1974, is a manifold equipped with two vector bundle structures which are compatible in a categorical sense, i.e., as a vector bundle in the category of vector bundles.

Definition

A double vector bundle (DVB in short) $(D; A, B; M)$ is a system of four vector bundle structures and VB-morphisms,

$$\begin{array}{ccc} D & \xrightarrow{q^D_B} & B \\ {\scriptstyle q^D_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M \end{array}$$

Moreover, each of the structure maps of each of the vector bundle structures on D (the bundle projection, the zero section, the addition, and the scalar multiplication) is a morphism of vector bundles with respect to the other structure.

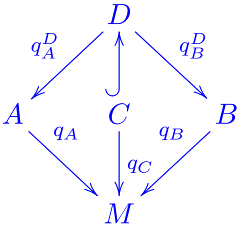
The core of a double vector bundle

Let C be the intersection of the two kernels:

$$C = \{c \in D \mid \exists\ m \in M \text{ such that } q^D_B(c) = 0^B_m, \quad q^D_A(c) = 0^A_m\}.$$

It is called the core of D and, together with the map $q_C(c) = m$, it is a vector bundle $q_C : C \to M$.

We can illustrate the core in the diagram of the double vector bundle as follows.



One can prove that

$$(q^D_A, q^D_B) : D \to A \oplus_M B$$

is an affine bundle modelled on the vector bundle $(q_A, q_B)^*(C)$.

Double vector bundles - the reference example

Let $q_A : A \to M$, $q_B : B \to M$, $q_C : C \to M$ be vector bundles. Consider the manifold

$$D = A \times_M B \times_M C\,.$$

Then D is canonically a double vector bundle, with the side bundles A and B, the core C, and with the obvious projections,

$$q^D_A : A \times_M B \times_M C \to A\,, \quad q^D_B : A \times_M B \times_M C \to B\,,$$

the obvious zero-sections,

$$\tilde{0}^A : A \ni a_m \mapsto (a_m, 0^B_m, 0^C_m) \in D\,,$$
$$\tilde{0}^B : B \ni b_m \mapsto (0^A_m, b_m, 0^C_m) \in D\,,$$

and the obvious vector space structures in fibers.

Actually, every double vector bundle is locally of this form.

In particular, the Whitney sum $A \oplus_M B$ is a double vector bundle with the trivial core.

Double Graded Bundles

Our understanding of vector bundles as homogeneity structures extremely simplifies the ‘categorical’ definition of Pradines.

Theorem (Grabowski-Rotkiewicz 2009)

Two vector bundle structures, $q^D_A : D \to A$ and $q^D_B : D \to B$, on a manifold D are compatible if and only if the corresponding homogeneity structures commute:

$$h^A_t \circ h^B_s = h^B_s \circ h^A_t \quad \textit{for all} \quad t, s \in \mathbb{R}.$$

Definition

A double graded bundle (DGB) is a manifold D equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all} \quad t, s \in \mathbb{R}\,.$$

If degrees of h^1, h^2 are k, l, then (k, l) we call the bi-degree of D.

Of course, all this can be naturally generalized to a concept of a n-fold graded bundle of n-degree $(k_1, \ldots, k_n)$.

Tangent lifts of graded bundles

The compatibility condition can also be formulated as the commutation of the weight vector fields, $[\nabla^1, \nabla^2] = 0$. Note that $h_t^{\mathsf{tot}} = h_t^1 \circ h_t^2$ defines the total graded bundle structure of D. An extension to n-tuple graded bundles is obvious.

Let $\tau : F \to M$ be a graded bundle of degree k with the homogeneity structure h, which in local coordinates (x^A, y_w^a), $1 \le w \le k$, reads $h_t(x^A, y_w^a) = (x^A, t^w y_w^a)$.

Let us consider the tangent lift $(\mathrm{d}_{\mathsf{T}} h)_t = \mathsf{T} h_t$ of h, i.e.,

$$(\mathrm{d}_{\mathsf{T}} h)_t(x^A, y_w^a, \dot{x}^B, \dot{y}_w^b) = (x^A, t^w y_w^a, \dot{x}^B, t^w \dot{y}_w^b)\,.$$

Proposition

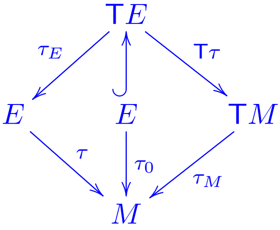
The tangent lift $\mathrm{d}_{\mathsf{T}} h$ is a homogeneity structure of degree k on $\mathsf{T}F$, which is compatible with the canonical vector bundle structure on $\mathsf{T}F$. In other words, the tangent bundle of a graded bundle is canonically a double graded bundle.

Canonical example

$\tau : E \longrightarrow M$ $\quad$ $(x^a, y^i) \mapsto (x^a)$

$\tau_E : \mathsf{T}E \longrightarrow E$ $\quad$ $(x^a, y^i, \dot{x}^b, \dot{y}^j) \mapsto (x^a, y^i)$

$\mathsf{T}\tau : \mathsf{T}E \longrightarrow \mathsf{T}M$ $\quad$ $(x^a, y^i, \dot{x}^b, \dot{y}^j) \mapsto (x^a, \dot{x}^b)$



$$h_t(x^a, y^i) = (x^a, ty^i), \quad (\mathrm{d}_{\mathsf{T}}h)_t(x^a, y^i, \dot{x}^b, \dot{y}^j) = (x^a, ty^i, \dot{x}^b, t\dot{y}^j)$$

$$\nabla = \textstyle\sum_i y^i \partial_{y^i}, \quad \nabla^1 = \mathrm{d}_{\mathsf{T}}(\sum_i y^i \partial_{y^i}) = \sum_i \left(y^i \partial_{y^i} + \dot{y}^i \partial_{\dot{y}^i}\right)$$
$$\nabla^2 = \textstyle\sum_a \dot{x}^a \partial_{\dot{x}^a} + \sum_i \dot{y}^i \partial_{\dot{y}^i}, \quad [\nabla^1, \nabla^2] = 0.$$

In the case $E = \mathsf{T}M$, there is a canonical automorphism (called the flip) $\kappa : \mathsf{TT}M \to \mathsf{TT}M$, intertwining both VB-structures.

Break

I INVITE YOU TO TAKE A BREAK

Phase lifts of graded bundles

To lift h_t from a graded bundle $\tau : F \to M$ of rank k to the cotangent bundle T^*F, consider the adapted coordinates (x^A, y^a_w, p_B, p^w_b) and

$$(\mathsf{T}h_{t^{-1}})^*(x^A, y^a_w, p_B, p^w_b) = (x^A, t^w y^a_w, p_B, t^{-w} p^w_b)$$

for $t \neq 0$. This makes no sense for $t = 0$, so we define the phase lift of h by $(\mathrm{d}_{\mathsf{T}^*} h)_t = t^k \cdot \left(\mathsf{T}h_{t^{-1}}\right)^*$. This makes sense for $t = 0$:

$$(\mathrm{d}_{\mathsf{T}^*} h)_t(x^A, y^a_w, p_B, p^w_b) = (x^A, t^w y^a_w, t^k p_B, t^{k-w} p^w_b)\,.$$

Proposition

The phase lift $\mathrm{d}_{\mathsf{T}^*} h$ is a homogeneity structure on T^*F, which is compatible with the canonical vector bundle structure.
In other words, the cotangent bundle of a graded bundle is canonically a double graded bundle.

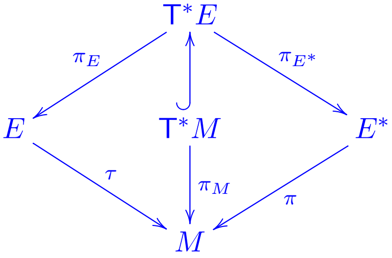
Phase lifts of vector bundles

$$\tau : E \longrightarrow M \qquad\qquad \pi_E : \mathsf{T}^*E \longrightarrow E$$
$$(x^a, y^i) \mapsto (x^a) \qquad\qquad (x^a, y^i, p_b, \xi_j) \mapsto (x^a, y^i)$$

$$h_t(x^a, y^i) = (x^a, ty^i), \quad (\mathrm{d}_{\mathsf{T}^*} h)_t(x^a, y^i, p_b, \xi_j) = (x^a, ty^i, tp_b, \xi_j)$$

The Poisson bracket $\{y^i, \xi_j\}$ is δ^i_j which implies that ξ_j are coordinates dual to y^i, so $(x^a, y^i, p_b, \xi_j) \mapsto (x^a, \xi_j)$ represents a projection $\zeta : \mathsf{T}^*E \to E^*$.

We have therefore a double vector bundle

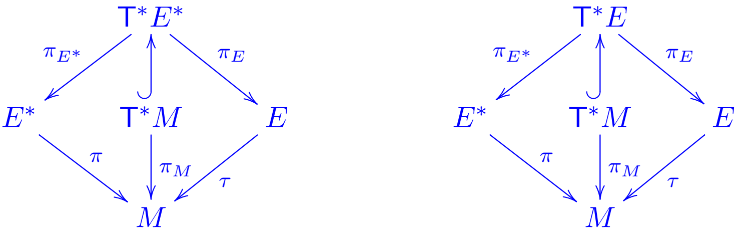


Linearity vs double vector bundles

Linearity of different geometrical structures is usually related to some DVB structures.

- A bivector field Λ on a vector bundle E is linear if the corresponding map
$$\Lambda^{\#} : \mathsf{T}^*E \longrightarrow \mathsf{T}E$$
is a morphism of double vector bundles.
- A two-form ω on a vector bundle E is linear if the corresponding map
$$\omega^{\flat} : \mathsf{T}E \longrightarrow \mathsf{T}^*E$$
is a morphism of double vector bundles.
- A distribution $D \subset \mathsf{T}E$ on a vector bundle E is linear if D is a double vector subbundle, i.e., D is a vector subbundle with respect to the both vector bundle structures on E.
- A (linear) connection on a vector bundle E is a horizontal distribution in $\mathsf{T}E$ (Ehresmann connection) which is linear.

Canonical isomorphism $\mathsf{T}^*E \simeq \mathsf{T}^*E^*$



Theorem (Tulczyjew 1974)

There is a canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^*E^* \to \mathsf{T}^*E$$

which in the adapted local coordinates reads

$$\mathcal{R}(x^a, \xi^i, p_b, \pi_j) = (x^a, \pi_i, -p_b, \xi^j)\,.$$

The map $\mathcal{R}$ is simultaneously an anti-symplectomorphism.

Canonical DVBs in mechanics

Let us put now $E = \mathsf{T}M$ to be the vector bundle of kinematical configurations. We know already that $\mathsf{T}^*\mathsf{T}M$ and $\mathsf{T}^*\mathsf{T}^*M$ are canonically DVBs which are canonically isomorphic:

$$\mathcal{R} : \mathsf{T}^*\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}M,$$
$$(q^i, p_j, \pi_k, y^l) \mapsto (q^i, y^j, -\pi_k, p_l) \,.$$

It is easy to see that the above isomorphism is simultaneously an anti-symplectomorphism. The canonical symplectic form $\omega_M = \mathrm{d}p_i \wedge \mathrm{d}q^i$ on T^*M induces a VB-isomorphism

$$\beta_M : \mathsf{T}\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}^*M,$$
$$(q^i, p_j, \dot{q}^k, \dot{p}_l) \mapsto (q^i, p_j, -\dot{p}_k, \dot{q}^l),$$

which is actually a DVB-isomorphism and anti-symplectomorphism with respect to the lifted symplectic structure

$$\mathrm{d}_{\mathsf{T}}(\omega_M) = \mathrm{d}\dot{p}_i \wedge \mathrm{d}q^i + \mathrm{d}p_i \wedge \mathrm{d}\dot{q}^i$$

on $\mathsf{T}\mathsf{T}^*M$.

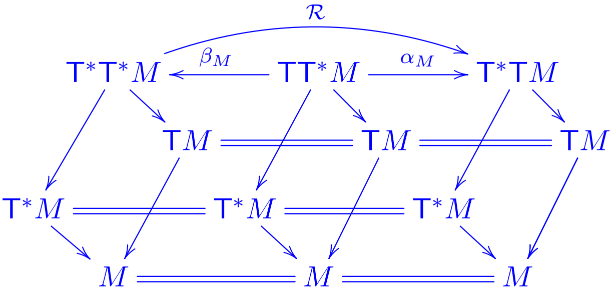
The Tulczyjew triple

Consequently,

$$\alpha_M = \mathcal{R} \circ \beta_M : \mathsf{T}\mathsf{T}^* M \to \mathsf{T}^*\mathsf{T} M,$$
$$(q^i, p_j, \dot{q}^k, \dot{p}_l) \mapsto (q^i, \dot{q}^j, \dot{p}_k, p_l)$$

is a DVB-isomorphism which is simultaneously a symplectomorphism. It is called the Tulczyjew isomorphism.

The full diagram of these symplectic DVB-isomorphisms, called the Tulczyjew triple, is the following:



Dynamics

The Tulczyjew's approach to formalism of mechanics uses the modern concept of first-order dynamics (first-order ODE), more general than the one based on just vector fields.

Definition

An implicit first-order dynamics on a manifold N is a submanifold $D \subset \mathsf{T}N$. A smooth curve $\gamma : \mathbb{R} \to N$ is a solution, if its tangent prolongation $\dot{\gamma} : \mathbb{R} \to \mathsf{T}N$ takes values in D.

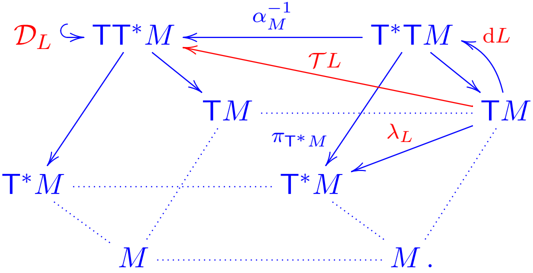
Example

A vector field X on N, defines the dynamics $D = X(N) \subset \mathsf{T}N$. Solutions for D are exactly trajectories of X.
Images of vector fields are exactly those submanifolds D of $\mathsf{T}M$ which are projected diffeomorphically on M by the bundle projection $\tau_M : \mathsf{T}M \to M$.

Similarly, submanifolds of T^2N are understood as (implicit) ordinary second-order differential equations, etc.

The Tulczyjew triple - the Lagrangian side

For a Lagrangian $L : \mathsf{T}M \to \mathbb{R}$, the phase dynamics $\mathcal{D}_L$ on T^*M is the image of the Tulczyjew differential $\mathcal{T}L = \alpha_M^{-1} \circ \mathrm{d}L$, called sometimes also the time evolution operator,



Dynamics $\mathcal{D}_L = \mathcal{T}L(\mathsf{T}M)$ is explicit for hyperregular Lagrangians only, i.e., when the Legendre map,

$$\lambda_L = \pi_{\mathsf{T}^*M} \circ \mathrm{d}L : \mathsf{T}M \to \mathsf{T}^*M, \quad \lambda_L(q, \dot{q}) = \left(q, \frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right),$$

is a diffeomorphism. Note that the dynamics has been obtained purely geometrically and no variational calculus has been used.

The Euler-Lagrange equations

In general, the implicit dynamics looks like

$$\mathcal{D}_L = (\alpha_M^{-1} \circ \mathrm{d}L)(\mathsf{T}M) = \left\{ (q, p, \dot{q}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{q}}, \quad \dot{p} = \frac{\partial L}{\partial q} \right\}.$$

The physically meaningful phase dynamics lives on the phase space $\mathsf{T}^* M$, however, one usually derives a second-order dynamics on M in the coordinate form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}}\Big) = \frac{\partial L}{\partial q}.$$

Note, however, that the information about momenta is lost in this passage. To derive the second-order equations geometrically, consider $\mathsf{T}\mathcal{T}L : \mathsf{TT}M \to \mathsf{TTT}^* M$ and take

$$\mathcal{D}_{EL} = (\mathsf{T}\mathcal{T}L)^{-1}(\mathsf{T}^2\mathsf{T}^* M) \subset \mathsf{T}^2 M,$$

where we view $\mathsf{T}^2 M$ as the submanifold of holonomic vectors in $\mathsf{TT}M$, i.e., fixed points of the canonical 'flip', $\dot{q} = \delta q$.

Euler-Lagrange equations (continued)

In local coordinates,

$$\mathcal{T}L(q,\dot q)=\left(q,\frac{\partial L}{\partial \dot q},\dot q,\frac{\partial L}{\partial q}(q,\dot q)\right),$$

so

$$\mathsf{T}\mathcal{T}L(q,\dot q,\delta q,\delta\dot q)=\Big(q,\frac{\partial L}{\partial \dot q},\dot q,\frac{\partial L}{\partial q},\delta q,\frac{\partial^2 L}{\partial \dot q^2}\delta\dot q+\frac{\partial^2 L}{\partial \dot q\,\partial q}\delta q\,,$$
$$\delta\dot q,\frac{\partial^2 L}{\partial \dot q\,\partial q}\delta\dot q+\frac{\partial^2 L}{\partial q^2}\delta q\Big)\,.$$

As holonomic vectors satisfy $\delta q=\dot q$ and $\delta p=\dot p$, we have

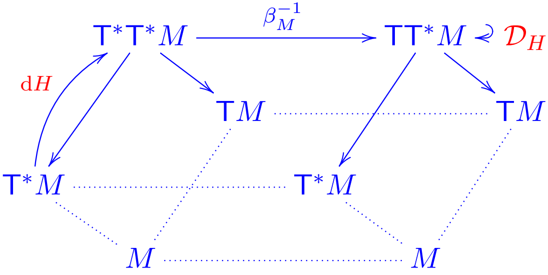
$$\mathcal{D}_{EL}=\left\{(q,\dot q,\delta q,\delta\dot q)\,\middle|\,\delta q=\dot q,\frac{\partial L}{\partial q}=\frac{\partial^2 L}{\partial \dot q^2}\delta\dot q+\frac{\partial^2 L}{\partial \dot q\,\partial q}\delta q\right\}.$$

Hence, $\mathcal{D}_{EL}\subset\mathsf{T}^2M$ and, interpreting $\delta\dot q$ as $\ddot q$, we get the Euler-Lagrange equations in the form

$$\frac{\partial^2 L}{\partial \dot q^2}\ddot q+\frac{\partial^2 L}{\partial \dot q\,\partial q}\dot q-\frac{\partial L}{\partial q}=0.$$

The Tulczyjew triple - the Hamiltonian side

For a Hamiltonian $H : \mathsf{T}^*M \to \mathbb{R}$, the phase dynamics $\mathcal{D}_H$ on T^*M is always explicit – the image of the Hamiltonian vector field $X_H = \beta_M^{-1} \circ \mathrm{d}H$:



We have then:

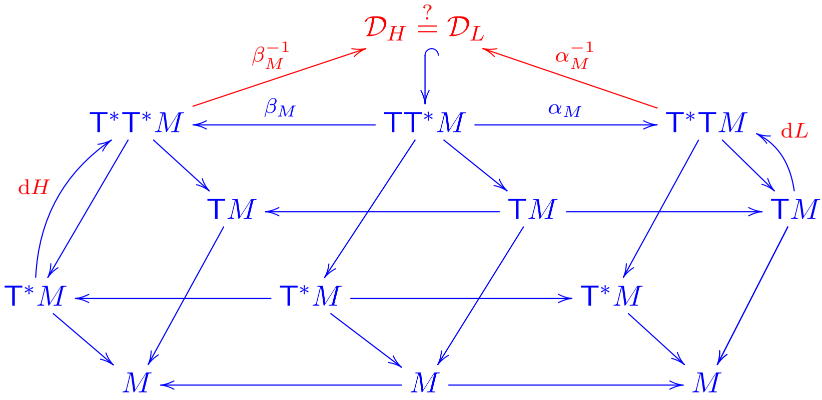
$$\mathcal{D}_H = \beta_M^{-1}\big(\mathrm{d}H(\mathsf{T}^*M)\big) = \left\{ (q,p,\dot q,\dot p) : \quad \dot p = -\frac{\partial H}{\partial q}, \quad \dot q = \frac{\partial H}{\partial p} \right\}.$$

Hence, the dynamics is described by the Hamilton equations.

Legendre transformation

The final picture is the following:

Hamiltonian side | phase dynamics | Lagrangian side



Note that $\mathcal{D}_H, \mathcal{D}_L, \mathrm{d}L(\mathsf{T}M), \mathrm{d}H(\mathsf{T}^*M)$ are always lagrangian submanifolds of the symplectic manifolds $\mathsf{T}\mathsf{T}^*M, \mathsf{T}^*\mathsf{T}M, \mathsf{T}\mathsf{T}^*M$, respectively.

The Legendre transformation

The Legendre transformation is a procedure of passing from a Lagrangian to a Hamiltonian description of the system. Generally, a Lagrangian description has a Hamiltonian formulation, i.e., $\mathcal{D}_L = \mathcal{D}_H$ for some Hamiltonian H, only for hyperregular Lagrangians, i.e., when the Legendre map $\lambda_L : \mathsf{T}M \to \mathsf{T}^*M$ is a diffeomorphism.

Thus, contrary to the belief of many physicists, the Lagrangian and Hamiltonian formalisms are generally not equivalent.

A way out is to consider not a single Hamiltonian but Morse families. It is well known that if the Lagrangian $L : \mathsf{T}M \to \mathbb{R}$ is hyperregular, then $\mathcal{D}_L = \mathcal{D}_H$ for the Hamiltonian function

$$H(q,p) = \dot{q}^i p_i - L, \quad \text{where} \quad (q,\dot{q}) = \lambda_L^{-1}(q,p).$$

In this case, the Lagrangian submanifolds $\mathrm{d}L(\mathsf{T}M) \subset \mathsf{T}^*\mathsf{T}M$ and $\mathrm{d}H(\mathsf{T}^*M) \subset \mathsf{T}^*\mathsf{T}^*M$ are related by $\mathcal{R} : \mathsf{T}^*\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}M$.

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THANK YOU FOR YOUR ATTENTION!

(Sokolica-Carpathians)