Introduction to $\mathbb{Z}-$graded manifolds

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- The talk is based on a series of papers publish jointly with Vladimir Salnikov (La Rochelle), Camille Laurent-Gengoux (Metz), Benoit Jubin (Luxembourg, Paris), and Norbert Poncin (Luxembourg).
- The general theory of $Z-$-graded manifolds was developed in

Kotov, A., Salnikov, V. The category of Z-graded manifolds: what happens if you do not stay positive, *Differential Geometry and its Applications* **93**, 102109 (2024)

- Classification theorems for smooth $Z-$-graded Q-manifolds are explained and proven in

Kotov, A., Laurent-Gengoux, C., Salnikov, V. Normal forms of $\mathbb{Z}$-graded Q-manifolds, *J. Geom. Phys.* **191**, 104908 (2023)

The last two articles in the list are devoted to graded and dg Lie (super) groups.

- Jubin, B., Kotov, A., Poncin, N., Salnikov, V. Differential graded Lie groups and their differential graded Lie algebras. *Transformation Groups* **27** 497–523 (2022)
- Kotov, A., Salnikov, V. Various instances of Harish-Chandra pairs, *J. Geom. Phys.* **191**, 104917 (2023)

A $\mathbb{Z}$-graded manifold is a topological space X together with a sheaf of $\mathbb{Z}-$graded algebras, locally modelled as free algebras of coordinates of degrees (weights) $\dots, -2, -1, 0, 1, 2$.

For example:

coordinates	ψ^I	ζ^k	x^i	ξ^a	η^α
weights	-2	-1	0	1	2

What is a free algebra in this context?

In the smooth case we require the free algebra to include:

- smooth functions of degree 0 coordinates;
- polynomials of coordinates of non-zero degree

The change of coordinates must preserve the degree!

For example,

$$x^i \longmapsto x^i + \xi^a \zeta^k + \psi^I \eta^\alpha, \;\; \xi^a \longmapsto \xi^a, \;\; \dots$$

where the coordinate weights are the same as above.

The change of coordinates:

$$x^i \longmapsto x^i + \xi^a \zeta^k + \psi^I \eta^\alpha, \;\; \xi^a \longmapsto \xi^a, \;\; \dots$$

The weight table:

coordinates	ψ^I	ζ^k	x^i	ξ^a	η^α
weights	-2	-1	0	1	2

Along with coordinate weights, we will also talk about their parities, which tell us how the coordinates commute.

- The parity takes values in $\mathbb{Z}_2$;
- Often the parity of a coordinate is the reduction of its weight modulo 2;
- But it is not the only possibility: we only require that, roughly speaking, the set of coordinates decomposed into the subsets of odd and even coordinates;
- odd coordinates anti-commute with each other (and thus their squares are equal to 0), while even coordinates commute with all other functions.

Let us compute the expression for $\cos(x^i)$:

$$\cos(x^i) \longmapsto \cos\left(x^i + \xi^a \zeta^k + \psi^l \eta^\alpha\right) =$$

$$= \text{Taylor expansion w.r.t. } \xi^a, \zeta^k, \psi^l, \eta^\alpha$$

- Using the described procedure we obtain polynomials of odd coordinates (since they are nilpotent).
- However, with even coordinates a "problem" arises: we have to assume a formal power series.

Another possibility, an alternative to formal series, is to consider smooth functions of coordinates not only of zero, but also of non-zero weight. Then we will arrive at another version of $\mathbb{Z}$-graded manifolds ("non-perturbative", speaking in the physical language of field theory). Although this approach is not the subject of the talk, a few words will be said about it later.

Formal power series in the $\mathbb{Z}-$graded context

Let $W = \bigoplus_{i \in \mathbb{Z}\setminus\{0\}} W_i$ be a $\mathbb{Z}$-graded vector space ($W_0 = 0$).

Let M be a smooth manifold, $U \subset M$ a coordinate chart with smooth coordinates $\{x^i\}$.

Let V be a free $C^\infty(U)-$module generated by W, i.e.

$$V = \bigoplus_i V_i$$

where

$$V_i = C^\infty(U) \otimes W_i$$

Define

$$\mathcal{A} = Sym_{C^\infty(U)}(V) = T_{C^\infty(U)}(V) \Big/ \langle v_1 \otimes v_2 - (-1)^{p_1 p_2} v_2 \otimes v_1 \rangle$$

where p_1, p_2 are the parities of v_1, v_2, respectively.

$\mathcal{A}$ is a $\mathbb{Z}-$graded supercommutative algebra (i.e. the elements of pure parity commute w.r.t. their parities):

$$\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$$

where $\mathcal{A}_i$ consists of elements of weight i (we apply the standard rule for the degree of the product of two elements - it's the sum of the degrees).

For any $p \geq 0$, let $F^p\mathcal{A}$ be the ideal of $\mathcal{A}$ generated by all elements of degree $\geq p$, that is,

$$F^p\mathcal{A} = \left\langle \bigoplus_{j \geq p} \mathcal{A}_j \right\rangle$$

We obtain a decreasing filtration of $\mathcal{A}$ by ideals

$$\mathcal{A} = F^0\mathcal{A} \supset F^1\mathcal{A} \supset F^2\mathcal{A} \supset F^3\mathcal{A} \supset \dots$$

For all $p \geq q$, there is a canonical morphism of $\mathbb{Z}-$graded supercommutative algebras

$$\mathcal{A}/F^p\mathcal{A} \xrightarrow{\pi_{p,q}} \mathcal{A}/F^q\mathcal{A}$$

We obtain a projective system of algebras

$$\left(\mathcal{A}/F^p\mathcal{A},\, \pi_{p,q}\right)$$

Let

$$\widehat{\mathcal{A}} = \varprojlim \mathcal{A}/F^p\mathcal{A}$$

be the projective (inverse) limit of this system. It is a $\mathbb{Z}-$graded supercommutative algebra.

Since

$$\mathcal{A}_k = \bigoplus_{i_1+\ldots+i_s=k} V_{i_1} \odot \ldots \odot V_{i_s}$$

even if each homogeneous component V_i has finite rank, the rank of $\mathcal{A}_k$ need not be finite!

Decompose

$$V = V_{+} \bigoplus V_{-}$$

where

$$V_{\pm} = \bigoplus_{\pm i > 0} V_i$$

is the sum of elements of positive (negative) weight, respectively.

Define $w_{\pm}$ the positive and negative degree, such that

$$w = w_{+} - w_{-}$$

where w_{+} and w_{-} count "positive" and "negative" degrees independently.

For example

coordinates	ψ^I	ζ^k	ξ^a	η^α
w_+	0	0	1	2
w_-	2	1	0	0
$w = w_+ - w_-$	-2	-1	1	2

For

$$f(x,\xi,\eta,\psi) = f_+(x,\xi,\eta) f_-(x,\zeta,\psi)$$

one has

$$w_+ f = w(f_+) \quad w_- f = -w(f_-)$$

Consider $\left(\mathcal{A}/F^p\mathcal{A}\right)_i$ for some i.

$\left(\mathcal{A}/F^p\mathcal{A}\right)_i$ has finite rank since $w_+ < p$

- $\left(\mathcal{A}/F^p\mathcal{A}\right)_i$ is a trivial $C^\infty(U)$ module of finite graded rank (i.e. the rank of each homogeneous component is finite);
- the space of its sections is a sheaf over U;
- each $\mathcal{A}_i = \varprojlim \left(\mathcal{A}/F^p\mathcal{A}\right)_i$ is a sheaf over U;
- $\mathcal{A}$ is a sheaf over U.

Global theory

Let M be a smooth manifold.

- Construct a presheaf of $\mathbb{Z}-$graded supercommutative algebras locally modelled by $\widehat{Sym}_{C^\infty(U)}(V)$;
- It is a sheaf since it is obtained by gluing sheaves over an open covering of M;
- Section of this sheaf are called functions on a $\mathbb{Z}-$graded smooth manifold $\mathcal{M}$ with the base M:

$$\widehat{\mathcal{A}} = \mathcal{F}(\mathcal{M})$$

$\mathbb{N}-$graded

We can allow polynomials of positive degree variables whose coefficients are smooth functions of zero degree variables

$\mathcal{F}_0(\mathcal{M}) = C^\infty(M)$ - this means that there is a canonical projection $\mathcal{M} \to M$

There is a canonical section ("zero section") of the above projection map:

$$\mathcal{M} \to M, \; M \subset \mathcal{M}$$

$\mathbb{Z}-$graded

This class of local functions is not stable under the homogeneous change of coordinates

$\mathcal{F}_0(\mathcal{M}) \neq C^\infty(M)$ - there is no canonical projection of a general $\mathbb{Z}-$graded smooth manifold onto its base

There is a canonical embedding $M \subset \mathcal{M}$: Let I be an ideal generated by all functions of non-zero weight. Then

$$C^\infty(M) = \mathcal{F}(\mathcal{M})/I$$

Supermanifolds: reminder

Let $E \to M$ be a vector bundle over a manifold (smooth, analytic, algebraic). We associate to it a $\mathbb{Z}_2-$graded (or super) manifold $\mathcal{M} = \Pi E$, such that the algebra of functions on it is

$$\mathcal{F}(\mathcal{M}) = \Gamma_M(\Lambda^\bullet E^*)$$

A supermanifold is characterized by its structure sheaf of $\mathbb{Z}_2-$graded supercommutative algebras over a base M locally modelled by $\Gamma_U(\Lambda^\bullet E^*)$, where $U \subset M$ is an open chart and $E \to U$ is a vector bundle.

- At least some supermanifolds (smooth, algebraic, analytic) are constructed out of global vector bundles as above. They are called split supermanifolds;
- Let I be the ideal of the structure sheaf generated by all odd functions. Then

$$\mathcal{F}(M) = \mathcal{F}(\mathcal{M})/I$$

- The latter means that, as in the Z-graded case, there is a canonical embedding $M \subset \mathcal{M}$;
- Just as in the general $\mathbb{Z}-$graded case and unlike the $\mathbb{N}-$graded situation, there is no canonical projection $\mathcal{M} \to M$;
- We can canonically associate a split supermanifold to any supermanifold using the scheme we describe below.

Define a vector bundle E over M as follows:

$$\Gamma_M(E^*) = I/I^2$$

One can easily show that I/I^2 is a locally free sheaf of $\mathcal{F}(M) = \mathcal{F}(\mathcal{M})/I$ modules, thus we obtained a vector bundle over the base canonically associated to the supermanifold.

Hence there is another split supermanifold $\overline{\mathcal{M}}$ uniquely determined by $\mathcal{M}$.

Batchelor-Gawedzki theorem

In the smooth case

$$\mathcal{M} \simeq \overline{\mathcal{M}}$$

Back to $\mathbb{Z}-$graded manifolds

- We already know that

$$\mathcal{F}(M) = \mathcal{F}(\mathcal{M})/I$$

for the ideal I generated by all functions of non-zero weight.

- Define a $\mathbb{Z}-$graded vector bundle $E \to M$ using the same procedure as for a supermanifold:

$$\Gamma_M(E^*) = I/I^2$$

- Let $\overline{\mathcal{M}}$ be another $\mathbb{Z}-$graded manifold, such that

$$\mathcal{F}(\overline{\mathcal{M}}) = \Gamma_M\left(\widehat{Sym}(E^*)\right)$$

$\mathbb{Z}-$graded Batchelor-Gawedzki theorem

In the smooth case

$$\mathcal{M} \simeq \overline{\mathcal{M}}$$

"Non-perturbative" approach to $\mathbb{Z}-$graded manifolds

Let $\mathcal{M}$ be a $\mathbb{Z}-$graded manifold. Then the algebra of functions is a direct sum of its homogeneous components:

$$\mathcal{F}(\mathcal{M}) = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}_i(\mathcal{M})$$

Define an operator $\epsilon\colon \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$, such that

$$\epsilon|_{\mathcal{F}_i(\mathcal{M})} = i\,\mathrm{Id}_{\mathcal{F}_i(\mathcal{M})}$$

ϵ is a derivation of an even parity

This suggests another definition of a $\mathbb{Z}-$graded supermanifold (T. Voronov):

A $\mathbb{Z}-$graded manifold is a supermanifold endowed with an even derivation of functions (a vector field) ϵ and a local covering by coordinate charts $(U, \{z^i\})$, such that

$$\epsilon(z^i) = k_i z^i$$

for some integers k_i.

From the definition it follows that the zero locus M of ϵ is a smooth supermanifold. This is exactly the base of $\mathcal{M}$.

$\mathbb{Z}-$graded Borel lemma (Kotov, Salnikov)

For any formal (in the normal direction) power series f on M of weight k there exists a smooth function $\tilde{f}$ on $\mathcal{M}$ of the same weight k, the (normal) Taylor expansion of which is f.

Example

$\mathcal{M} = \mathbb{R}P^1$ with the standard affine coordinate charts $\left(U_0, z\right)$ and $\left(U_\infty, u\right)$ with the relation $z = \frac{1}{u}$ on the double overlap $U_0 \cap U_\infty$.

Define $\epsilon = z\frac{\partial}{\partial z}$ over U_0; this vector field uniquely extends to U_∞ as $\epsilon = -u\frac{\partial}{\partial u}$.

- The zero locus of ϵ is $\{0\} \cup \{\infty\}$. It is disconnected;
- The weights around the connected components of ϵ are different: it is $+1$ around 0 and -1 around ∞.

In general, the weights in a neighborhood of a connected component of the zero locus of ϵ are the same, while around different components they can be different. The formal neighborhood of a connected component is a $\mathbb{Z}-$graded manifold in the "previous" sense.

$\mathbb{Z}-$graded Lie algebras and Lie groups

A $\mathbb{Z}-$graded Lie algebra is a Lie superalgebra $\mathfrak{g}$ together with a grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

such that the Lie bracket $[,] \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is a degree 0 operation, i.e.

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

The corresponding grading operator ϵ, defined such that

$$\epsilon|_{\mathfrak{g}_i} = i \, \mathrm{Id}_{\mathfrak{g}_i}$$

is a degree 0 derivation of $\mathfrak{g}$, i.e. $\epsilon\big[x, y\big] = \big[\epsilon(x), y\big] + \big[x, \epsilon(y)\big]$

Let G be a Lie supergroup, such that its Lie superalgebra is $\mathfrak{g}$. Suppose ϵ_G be a vector field on G, the infinitesimal counterpart of which is ϵ.

This correspondence is an example of the van Est map. The obtained vector field is multiplicative, i.e. the multiplication map $G \times G \to G$ is $\epsilon-$equivariant.

Alternatively, one can illustrate it as follows: $\exp(t\epsilon_G)$, $t \in \mathbb{R}$ is a family of automorphisms of G, such that the induced family of automorphisms of $\mathfrak{g}$ is $\exp(t\epsilon)$.

A good definition of a $\mathbb{Z}$-graded Lie group should be such that it is the Lie group of a $\mathbb{Z}-$graded Lie algebra!

Does the pair (G, ϵ_G) satisfy the definition of a $\mathbb{Z}$-graded supermanifold?

It was shown (Kotov-Salnikov) that a vector field ϵ_G has the required form in a certain neighborhood of its zero locus, which is the Lie subgroup corresponding to $\mathfrak{g}_0$.

Question

Can we find an open covering of G by coordinate charts with adapted coodrinates (of pure weights w.r.t. ϵ_G)?

Q-manifolds

A Q-manifold (A. Schwarz) is a $\mathbb{Z}-$ graded supermanifold endowed with a homological degree 1 vector field. A Q-submanifold is a graded immersed super submanifold such that the corresponding immersion is a $Q-$morphism.

Examples of non-negatively graded Q-manifolds

- Lie algebroids (A. Vaintrob, Q-manifolds of degree 1); the Q-field is given by the Lichnerowicz differential
- For Lie algebras the Lichnerowicz differential is the the Chevalley-Eilenberg differential
- $T[1]M$ for a graded supermanifold; the Q-field is the de Rham operator
- Lie-infinity algebroids (general non-negatively or $\mathbb{N}-$graded Q-manifolds)

Examples of Q-manifolds

- L_∞−algebras (viewed as formal pointed Q-manifolds)
- symplectic Q-manifolds (graded super symplectic manifolds whose symplectic structure is invariant under Q)
- In particular, the symplectic degree 2 Q-manifold corresponding to a Courant algebroid (D. Roytenberg, A. Weinstein)
- The group-like objects in the category of Q-manifolds are dg or Q-groups (B. Jubin, A.K., N. Poncin, V. Salnikov, 2019-22, integration of dg Lie algebras to dg Lie groups)
- The differential graded resolution of a (possibly) singular variety, an example of a non-positively graded Q-manifold

Example of a differential graded (Koszul) resolution

Coordinates on the $\mathbb{Z}-$graded supermanifold are x, y and ξ of weights 0, 0 and -1, respectively.

The first two coordinates are even, while the third one is odd

The base is $M = \mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2\}$

The differential $\delta = xy\frac{\partial}{\partial \xi}$, i.e.

$$\delta\big(x\big) = \delta\big(y\big) = 0,\ \delta\big(\xi\big) = xy$$

The cohomology of the corresponding non-positively graded complex $\big(\mathcal{F}(\mathcal{M}), \delta\big)$ are concentrated in degree 0 and are equal to $\mathcal{F}(\mathbb{R}^2)/\langle xy\rangle$. We have obtained Koszul resolution of the singular set $xy = 0$.

Let $\mathcal{M}$ be a $\mathbb{Z}-$graded supermanifold, I_+ and I_- be the ideals of $\mathcal{F}(\mathcal{M})$ generated by functions of positive and negative weighs, respectively.

Define $\mathcal{M}_\pm \subset \mathcal{M}$, such that $\mathcal{F}(\mathcal{M}_\pm) = \mathcal{F}(\mathcal{M})/I_\mp$

$\mathcal{M}_+$ and $\mathcal{M}_-$ is non-negatively and non-positively graded, respectively.

Assume that $\left(\mathcal{M}, Q\right)$ is a $\mathbb{Z}-$graded Q-manifold.

$Q\left(I_+\right) \subset I_+$, thus $\left(\mathcal{M}_-, Q_-\right)$ is a Q-submanifold of $\mathcal{M}$, where Q_- is the restriction of Q onto $\mathcal{M}_-$

For an arbitrary $\left(\mathcal{M}, Q\right)$, $Q\left(I_-\right) \not\subset I_-$, hence, in general, $\mathcal{M}_+$ is a not Q-submanifold

Consider $\widehat{I_-} = I_- + Q(I_-)$. Then $\widehat{I_-}$ is stable under the action of Q and thus it defines a Q-submanifold (Σ_+, Q_+), where Q_+ is the restriction of Q onto Σ_+

Thereafter we will look only those Q-manifolds for which $(\mathcal{M}_-, Q_-)$ is acyclic in all negative degrees (Koszul-Tate type)

Perturbation theory

Let M be a smooth manifold, $X \subset M$ be a singular subvariety which admits Koszul-Tate resolution $(\mathcal{M}_-, \delta)$, i.e.

$$H^0(\mathcal{F}(\mathcal{M}_-), \delta) = \mathcal{F}(X),\ H^{<0}(\mathcal{F}(\mathcal{M}_-), \delta) = 0$$

Let $\mathcal{M}_+$ be a non-negatively graded manifold with a base M and Q_+ be a homological degree 1 vector field on $\Sigma_+ = \mathcal{M}_+|_X$. Then there exists a unique (up to an isomorphism) $Q-$manifold $(\mathcal{M}, Q)$, the non-positive part of which coincides with $(\mathcal{M}_-, \delta) =$ and the restriction of Q onto $\mathcal{M}_+|_X$ coincides with (Σ_+, Q_+).