The local classification of finite-dimensional Lie algebras of analytic Hamiltonian vector fields on the plane

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Introduction

- Extension of Galois theory to differential equations,
- Continuous groups of transformations (Lie groups actions),
- Infinitesimal continuous groups of transformations (Lie algebras of vector fields).
- Families of first-order differential equations whose solutions are functions of a particular family of solutions and several parameters (Lie systems).

Lie obtained very important and relevant results, but his works were not clear, they lacked precision, and contained obscure statements that led misunderstandings.

Lie classified finite-dimensional Lie algebras of analytic vector fields:

- On $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^2$, $\mathbb{C}^2$, and $\mathbb{C}^3$ (this latter one partially).
- Several technical assumptions were used, and they are relevant.

Every first-order system of ordinary differential equations in normal form on an n-dimensional manifold amounts to a t-dependent vector field.

$$X(t,x)=\sum_{i=1}^n X^i(t,x)\frac{\partial}{\partial x^i} \Longleftrightarrow \frac{dx^i}{dt}=X^i(t,x), \quad i=1,\ldots,n.$$

Linear nonautonomous systems of ordinary differential equations admit linear superposition rules. The equation

$$\frac{dx}{dt}=a_0(t)+a_1(t)x+a_2(t)x^2, \qquad x\in\mathbb{R},$$

for arbitrary t-dependent functions $a_0(t), a_1(t), a_2(t)$, admits a nonlinear superposition rule

$$x(t)=\frac{x_1(t)(x_2(t)-x_3(t))-kx_2(t)(x_3(t)-x_1(t))}{x_2(t)-x_3(t)-k(x_3(t)-x_1(t))}$$

in terms of three particular different solutions $x_1(t), x_2(t), x_3(t)$.

Definition

A *superposition rule* for a system X on a manifold N is a function $\Phi : N^m\times N\to N$, $x=\Phi(x_{(1)},\ldots,x_{(m)};k)$, such that the general solution $x(t)$ of our system can be brought into the form

$$x(t)=\Phi(x_{(1)}(t),\ldots,x_{(m)}(t);k),$$

where $x_{(1)}(t),\ldots,x_{(m)}(t)$ is any generic family of particular solutions and k is a point of N to be related to initial conditions.

Vector fields and differential equations

Proposition (Guldberg & Königsberger up to 1893)

Every non-autonomous analytic first-order differential equation on the real line admitting a superposition rule is a particular case of linear or Riccati equation.

Every differential equation of the form

$$\frac{dx}{dt} = X(t,x), \qquad t \in \mathbb{R}, \qquad x \in \mathbb{R},$$

for an analytic function $X(t,x)$ admitting a superposition rule is locally diffeomorphic around a generic point $x_0 \in \mathbb{R}$ to

$$\frac{dx}{dt} = c_0(t) + c_1(t)x + c_2(t)x^2, \qquad t \in \mathbb{R}, \qquad x \in \mathbb{R},$$

for some t-dependent functions $c_0(t), c_1(t), c_2(t)$.

The Lie–Scheffers Theorem

A system X on N admits a superposition rule if and only if $X = \sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha$ for a family of vector fields $X_1, \ldots, X_r$ on N spanning a finite-dimensional Lie algebra V of vector fields and certain t-dependent functions $b_1(t), \ldots, b_r(t)$. We call V a *Vessiot–Guldberg Lie algebra* for the Lie system.

Lie algebras of vector fields on the real line

Proposition (Lie 1880)

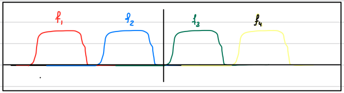
Every finite-dimensional Lie algebra of analytic vector fields on $\mathbb{R}$ admit an analytic coordinate system, around a point where one of its vector fields does not vanish, where it becomes a Lie subalgebra of

$$V_{\mathrm{Ric}} = \left\langle X_0 = \frac{\partial}{\partial x}, X_1 = x\frac{\partial}{\partial x}, X_2 = x^2\frac{\partial}{\partial x} \right\rangle \simeq \mathfrak{sl}_2$$

$$[X_0, X_1] = X_0, \qquad [X_0, X_2] = 2X_1, \qquad [X_1, X_2] = X_2.$$

Removing technical conditions is very interesting. For bump functions $f_i(x)$ as below:

$$X_i = f_i(x)\frac{\partial}{\partial x}, \qquad [X_i, X_j] = 0,$$
$$i = 1, 2, 3, 4, \ldots$$



Note that $X_k = y^k \frac{\partial}{\partial y}$, for $k > 2$, is not (even smooth) diffeomorphic to any element of V_{Ric} around $y = 0$:

$$X_k = y(x)^k \frac{dx}{dy}(y(x))\frac{\partial}{\partial x}, \qquad y(0) = 0, \qquad \frac{dy}{dx}(0) \neq 0,$$

since the first non-zero coefficient of the Taylor expansion in x is k (Hitchin's classification result).

Problems with Lie's classifications

Excerpt from A. González-López, N. Kamran, and P.J. Olver, *Lie algebras of vector fields in the real plane*, Proc. London Math. Soc. **64**, 339–368 (1992).

> These results will, we hope, clear up some confusing and contradictory statements in the literature on this problem. The classical authors, for example, Lie, Campbell, Bianchi, etc., never really made it clear whether they were working over the real or the complex numbers, which has led to confusion among more recent authors. For example, in their translation and commentary on Lie's paper [**9**], Hermann and Ackerman assert that the lemma on triangularization of matrices holds only over the complex numbers [**9**, p. 296], while the commentary is clearly aimed at real vector fields. A similar mis-statement occurs in Bluman and Kumei's recent book on symmetry groups of differential equations [**3**, p. 129], where it is falsely asserted that Campbell [**5**] obtained the classification of vector fields over $\mathbb{R}^2$. Bluman and Kumei go on to use Campbell's complex classification

The modern local classification of Lie algebras of analytic vector fields on the plane is called the GKO classification.

Finite-dimensional Lie algebras of analytic vector fields on the plane I

If not otherwise stated, Lie algebras of vector fields are finite-dimensional and analytic. Generalised distributions are called distributions. Hereafter, N is a manifold.

Definition

Given a Lie algebra V of vector fields on N, its *associated distribution* is the distribution $\mathcal{D}^V$ on N spanned by the vector fields of N, i.e.

$$\mathcal{D}^V_x = \langle X(x) | X \in V \rangle, \qquad \forall x \in N.$$

A distribution $\mathcal{D}^V$ is involutive and integrable, but not necessarily regular.

Definition

A *regular point* of a Lie algebra V of vector fields on N is a point where the associated distribution $\mathcal{D}$ is locally regular. If a point is not regular, it is called *singular*.

For instance, a vector field X on N with an isolated zero at $x_0 \in N$ spans a one-dimensional Lie algebra V_X, but its associated distribution, $\mathcal{D}^{V_X}$, has a singular point at $x_0 \in N$.

Definition

A Lie algebra V of vector fields on $\mathbb{R}^2$ is *imprimitive* when there exists a rank-one distribution $\mathcal{D} \subset T\mathbb{R}^2$ such that $[X, Y]$ takes values in $\mathcal{D}$ for every $X \in V$ and every vector field Y taking values in $\mathcal{D}$. Otherwise, V is called *primitive*.

Finite-dimensional Lie algebras of analytic vector fields on the plane II

Table 1: The GKO classification of the 8 finite-dimensional real primitive Lie algebras of vector fields on the plane. Note that $\mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ means that $\mathfrak{g}$ is the direct sum (as linear subspaces) of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, with $\mathfrak{g}_2$ being an ideal of $\mathfrak{g}$.

#	Primitive	Basis of vector fields X_i	Domain
P_1	$A_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, \alpha(x\partial_x + y\partial_y) + y\partial_x - x\partial_y, \quad \alpha \geq 0$	$\mathbb{R}^2$
P_2	$\mathfrak{sl}(2)$	$\partial_x, x\partial_x + y\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y$	$\mathbb{R}^2_{y\neq 0}$
P_3	$\mathfrak{so}(3)$	$y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y, 2xy\partial_x + (1 + y^2 - x^2)\partial_y$	$\mathbb{R}^2$
P_4	$\mathbb{R}^2 \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y$	$\mathbb{R}^2$
P_5	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_y$	$\mathbb{R}^2$
P_6	$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y$	$\mathbb{R}^2$
P_7	$\mathfrak{so}(3,1)$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y, 2xy\partial_x + (y^2 - x^2)\partial_y$	$\mathbb{R}^2$
P_8	$\mathfrak{sl}(3)$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y$	$\mathbb{R}^2$

Finite-dimensional Lie algebras of analytic vector fields on the plane III

Table 2: The GKO classification of the 20 finite-dimensional real imprimitive Lie algebras of vector fields.

#	Imprimitive	Basis of vector fields X_i	Domain
I_1	$\mathbb{R}$	∂_x	$\mathbb{R}^2$
I_2	$\mathfrak{h}_2$	$\partial_x, x\partial_x$	$\mathbb{R}^2$
I_3	$\mathfrak{sl}(2)$ (type I)	$\partial_x, x\partial_x, x^2\partial_x$	$\mathbb{R}^2$
I_4	$\mathfrak{sl}(2)$ (type II)	$\partial_x+\partial_y, x\partial_x+y\partial_y, x^2\partial_x+y^2\partial_y$	$\mathbb{R}^2_{x\neq y}$
I_5	$\mathfrak{sl}(2)$ (type III)	$\partial_x, 2x\partial_x+y\partial_y, x^2\partial_x+xy\partial_y$	$\mathbb{R}^2_{y\neq 0}$
I_6	$\mathfrak{gl}(2)$ (type I)	$\partial_x, \partial_y, x\partial_x, x^2\partial_x$	$\mathbb{R}^2$
I_7	$\mathfrak{gl}(2)$ (type II)	$\partial_x, y\partial_y, x\partial_x, x^2\partial_x+xy\partial_y$	$\mathbb{R}^2_{y\neq 0}$
I_8	$B_\alpha \simeq \mathbb{R}\ltimes\mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x+\alpha y\partial_y, \quad 0<\lvert\alpha\rvert\leq 1$	$\mathbb{R}^2$
I_9	$\mathfrak{h}_2\oplus\mathfrak{h}_2$	$\partial_x, \partial_y, x\partial_x, y\partial_y$	$\mathbb{R}^2$
I_{10}	$\mathfrak{sl}(2)\oplus\mathfrak{h}_2$	$\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x$	$\mathbb{R}^2$
I_{11}	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$	$\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x, y^2\partial_y$	$\mathbb{R}^2$
I_{12}	$\mathbb{R}^{r+1}$	$\partial_y, \xi_1(x)\partial_y, \ldots, \xi_r(x)\partial_y, \quad r\geq 1$	$\mathbb{R}^2$
I_{13}	$\mathbb{R}\ltimes\mathbb{R}^{r+1}$	$\partial_y, y\partial_y, \xi_1(x)\partial_y, \ldots, \xi_r(x)\partial_y, \quad r\geq 1$	$\mathbb{R}^2$
I_{14}	$\mathbb{R}\ltimes\mathbb{R}^r$	$\partial_x, \eta_1(x)\partial_y, \eta_2(x)\partial_y, \ldots, \eta_r(x)\partial_y, \quad r\geq 1$	$\mathbb{R}^2$
I_{15}	$\mathbb{R}^2\ltimes\mathbb{R}^r$	$\partial_x, y\partial_y, \eta_1(x)\partial_y, \ldots, \eta_r(x)\partial_y, \quad r\geq 1$	$\mathbb{R}^2$
I_{16}	$C_\alpha^r \simeq \mathfrak{h}_2\ltimes\mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x+\alpha y\partial y, x\partial_y, \ldots, x^r\partial_y, \quad r\geq 1, \quad \alpha\in\mathbb{R}$	$\mathbb{R}^2$
I_{17}	$\mathbb{R}\ltimes(\mathbb{R}\ltimes\mathbb{R}^r)$	$\partial_x, \partial_y, x\partial_x+(ry+x^r)\partial_y, x\partial_y, \ldots, x^{r-1}\partial_y, \quad r\geq 1$	$\mathbb{R}^2$
I_{18}	$(\mathfrak{h}_2\oplus\mathbb{R})\ltimes\mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_y, \ldots, x^r\partial_y, \quad r\geq 1$	$\mathbb{R}^2$
I_{19}	$\mathfrak{sl}(2)\ltimes\mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_y, 2x\partial_x+ry\partial_y, x^2\partial_x+rxy\partial_y, x^2\partial_y, \ldots, x^r\partial_y, \quad r\geq 1$	$\mathbb{R}^2$
I_{20}	$\mathfrak{gl}(2)\ltimes\mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_x+rxy\partial_y, x^2\partial_y, \ldots, x^r\partial_y, \quad r\geq 1$	$\mathbb{R}^2$

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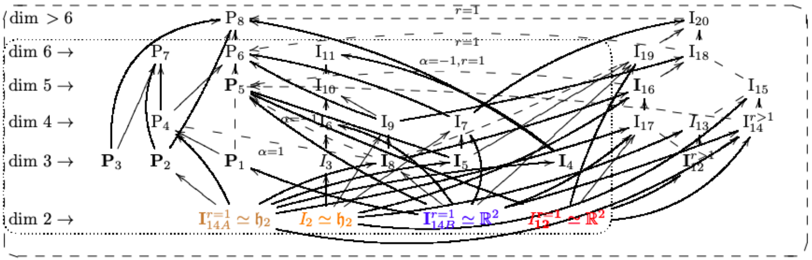
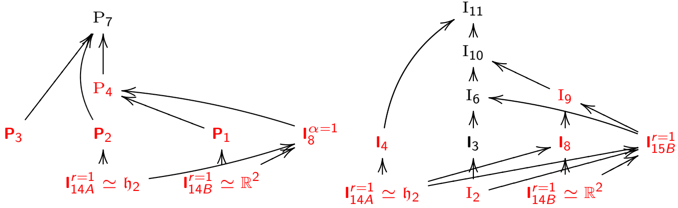
Relation between Lie algebras of analytic vector fields on the plane

A. Ballesteros, A. Blasco, F. J. Herranz, J. de Lucas, C. Sardón, *Lie-Hamilton systems on the plane: properties, classification and applications*, J. Differential Equations **258**, 2873–2907 (2015).

A.M. Grundland and J. de Lucas, *A Lie systems approach to the Riccati hierarchy and partial differential equations*, J. Differential Equations **263**, 299–337 (2017).

Table 3: Non-exhaustive tree of inclusion relations between classes of the GKO classification. The diagram details all Lie subalgebras of I_{11} and P_7. We write $A \to B$ when a subclass of A is diffeomorphic to a Lie subalgebra of B. Every Lie algebra includes I_1. In bold and italics are classes with Hamiltonian Lie algebras and rank one associated distributions, respectively. Colors help to distinguish the arrows.





Proposition

Every Lie algebra of conformally Euclidean (resp. hyperbolic) vector fields on $\mathbb{R}^2$ is locally diffeomorphic to a Lie subalgebra of P_7 (resp. I_{11}).

We have the following conjecture (partially proved). Recall that projective vector fields of a metric are those whose flow map geodesic into geodesics up to reparametrisation.

Conjecture

Every Lie algebra of projective vector fields on $\mathbb{R}^2$ relative to a flat Riemannian metric (resp. flat (1,1) pseudo-Riemannian) is locally diffeomorphic to a Lie subalgebra of P_8 (resp. I_{20}).

The above summarises all the GKO classification apart from I_{12} and I_{13}, but they can also be geometrically described as symmetries of degenerate two-contravariant symmetric tensors.

Symplectic manifolds

Definition

A *symplectic manifold* (N, ω) is a manifold N along with a closed non-degenerate two-form ω on N, whether non-degenerate amounts to the existence of $n \in \mathbb{N}$ such that ω^n is a volume form on N.

Definition

We say that X is a *Hamiltonian vector field* if $\iota_X\omega = df$ for a certain function $f \in C^\infty(N)$. We say that f is a Hamiltonian function for X. Conversely, each f induces a unique *Hamiltonian vector field* X_f.

Proposition

The space $\mathrm{Ham}(N, \omega)$ of Hamiltonian vector fields for a symplectic manifold (N, ω) is a Lie algebra.

Proposition

Every symplectic manifold (N, ω) induces a Poisson bracket $(C^\infty(N), \cdot, \{\cdot, \cdot\})$ of the form

$$\{f, g\} = \omega(X_f, X_g) = X_g f, \qquad \forall f, g \in C^\infty(N).$$

It can be proved that $X_{\{f,g\}} = -[X_f, X_g]$.

Proposition

A symplectic manifold (N, ω) induces an exact sequence of Lie algebras

$$0 \hookrightarrow \mathbb{R} \hookrightarrow C^\infty(N) \xrightarrow{B^\omega} \mathrm{Ham}(N, \omega) \to 0,$$

where $B^\omega(f) := -X_f$ for every $f \in C^\infty(N)$.

Poisson manifolds

Definition

A *Poisson manifold* $(N, \{\cdot,\cdot\})$ is a manifold N along with a Poisson bracket $\{\cdot,\cdot\} : C^\infty(N) \times C^\infty(N) \to C^\infty(N)$. Such a Poisson bracket is called a *Poisson structure*.

The Poisson structure induces a bivector $\Lambda \in \Gamma(\bigwedge^2 TN)$ such that

$$\{f, g\} = \Lambda(df, dg).$$

The Jacobi identity is equivalent to the condition $[\Lambda, \Lambda]_{SB} = 0$, where $[\Lambda, \Lambda]_{SB}$ is the *Schouten-Nijenhuis bracket*. Then, we can refer to Poisson manifolds as $(N, \{\cdot,\cdot\})$ or (N, Λ).

Definition

We say that X is a *Hamiltonian vector field* if there exists an $f \in C^\infty(N)$ such that $Xg = \{f, g\}$ for every $g \in C^\infty(N)$. We call f a *Hamiltonian function* for X.

Proposition

The space $\mathrm{Ham}(N, \Lambda)$ of Hamiltonian functions for a Poisson manifold (N, Λ) is a Lie algebra.

Proposition

We say that f is a *Casimir function* of $(N, \{\cdot, \cdot\})$ if $X_f = 0$. We write $\mathrm{Cas}(N, \Lambda)$ for the space of Casimir functions for (N, Λ).

Proposition

Every Poisson manifold (N, ω) induces an exact sequence of Lie algebras

$$0 \hookrightarrow \mathrm{Cas}(N, \Lambda) \hookrightarrow C^\infty(N) \xrightarrow{B^\Lambda} \mathrm{Ham}(N, \Lambda) \to 0,$$

where $B^\Lambda(f) = -X_f$ for every $f \in C^\infty(N)$.

Hamiltonian Lie systems on the plane

Consider the nonautonomous complex Bernoulli differential equations of the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = a_1(t)z + a_2(t)z^n, \qquad n \notin \{0,1\}, \qquad z \in \mathbb{C}, \tag{1}$$

where $a_1(t) = \mathrm{i}a_1^I(t)$ and $a_2(t) = a_2^R(t) + \mathrm{i}a_2^I(t)$ for real t-dependent functions $a_1^I(t), a_2^R(t), a_2^I(t)$. Writing $z = r\mathrm{e}^{\mathrm{i}\theta}$, system (1) becomes

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= a_2^R(t)\, r^n \cos[\theta(n-1)] - a_2^I(t)\, r^n \sin[\theta(n-1)], \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= a_1^I(t) + a_2^R(t)\, r^{n-1} \sin[\theta(n-1)] + a_2^I(t)\, r^{n-1} \cos[\theta(n-1)]. \end{aligned} \tag{2}$$

This system is related to $X = a_1^I(t)X_1 + a_2^R(t)X_2 + a_2^I(t)X_3$, where

$$\begin{gathered} X_1 = \frac{\partial}{\partial\theta}, \qquad X_2 = r^n \cos[\theta(n-1)]\frac{\partial}{\partial r} + r^{n-1}\sin[\theta(n-1)]\frac{\partial}{\partial\theta}, \\ X_3 = -r^n \sin[\theta(n-1)]\frac{\partial}{\partial r} + r^{n-1}\cos[\theta(n-1)]\frac{\partial}{\partial\theta} \end{gathered} \tag{3}$$

span Lie algebra, $V^{\mathrm{CB}} \simeq \mathbb{R} \ltimes \mathbb{R}^2 \simeq \langle X_1\rangle \ltimes \langle X_2, X_3\rangle$, with commutation relations

$$[X_1, X_2] = (n-1)X_3, \qquad [X_1, X_3] = -(n-1)X_2, \qquad [X_2, X_3] = 0. \tag{4}$$

So, X takes values in V^{CB} and becomes a Lie system. Since $X_1 \wedge X_2 \neq 0$ and $\mathrm{ad}_{X_1} : X_i \in \langle X_2, X_3\rangle \mapsto [X_1, X_i] \in \langle X_2, X_3\rangle$ is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$, then $V^{\mathrm{CB}} \simeq \mathrm{P}_1 \simeq \mathfrak{iso}(2)$.

Consider the Poisson bivector

$$\Lambda = r^{2n-1}\frac{\partial}{\partial r}\wedge\frac{\partial}{\partial\theta} \tag{5}$$

turning the elements of $V=\langle X_1,X_2,X_3\rangle$ into Hamiltonian vector fields. Indeed, some Hamiltonian functions for X_1, X_2, X_3 read

$$h_1=\frac{1}{(2n-2)r^{2n-2}},\qquad h_2=\frac{\sin[\theta(n-1)]}{r^{n-1}(n-1)},\qquad h_3=\frac{\cos[\theta(n-1)]}{r^{n-1}(n-1)},$$

correspondingly. These functions along with $h_0=1$ fulfill

$$\{h_1,h_2\}_\Lambda=-(n-1)h_3,\quad \{h_1,h_3\}_\Lambda=(n-1)h_2,\quad \{h_2,h_3\}_\Lambda=h_0,\quad \{h_0,\cdot\}_\Lambda=0.$$

Hence, system (2) with $a_1^R(t)=0$ is a LH system as it is related to a t-dependent vector field taking values in a Vessiot–Guldberg Lie algebra V of Hamiltonian vector fields relative to Λ. Meanwhile, the LH algebra spanned by h_1,h_2,h_3,h_0 is isomorphic to the centrally extended Euclidean algebra $\overline{\mathfrak{iso}}(2)$.

Relevant property

$$\Lambda=X_2\wedge X_3\qquad\Longrightarrow\qquad \mathcal{L}_{X_i}\Lambda=0,\qquad [\Lambda,\Lambda]_{SN}=0.$$

It is worth noting that the initial system can be studied as holomorphic. This simplifies some expressions, like (2), and make more complicated others, like (5), but the final idea is independent of the coordinate description.

Turning a Lie algebra of vector fields into Hamiltonian ones relative to a symplectic form involves solving a system of PDEs.

Definition

Given a vector space V of vector fields on U, we say that V admits a modular generating system $(U_1, X_1, \ldots, X_p)$ if U_1 is a dense open subset of U such that every $X \in V|_{U_1}$ can be brought into the form $X|_{U_1} = \sum_{i=1}^{p} g_i X_i|_{U_1}$ for certain functions $g_1, \ldots, g_p \in C^\infty(U_1)$ and vector fields $X_1, \ldots, X_p \in V$.

Theorem

Let V be a Lie algebra of vector fields on $U \subset \mathbb{R}^2$ admitting a modular generating system $(U_1, X_1, \ldots, X_p)$. We have that the space V consists of Hamiltonian vector fields relative to a symplectic form on U if and only if:

- *Let $g_1, \ldots, g_p$ be certain smooth functions on $U_1 \subset U$. Then,*

$$X|_{U_1} = \sum_{i=1}^{p} g_i X_i|_{U_1} \in V|_{U_1} \Rightarrow \mathrm{div} X|_{U_1} = \sum_{i=1}^{p} g_i \mathrm{div} X_i|_{U_1},$$

  where $\mathrm{div} X$ is relative to $\mathrm{dx} \wedge \mathrm{dy}$.
- *The elements $X_1, \ldots, X_p$ admit a common non-vanishing integrating factor on U.*

If the rank of D^V is two on U, the symplectic form is unique up to a multiplicative non-zero constant.

Proposition

Let V be a Lie algebra of planar vector fields. The vector fields of V are Hamiltonian with respect to a bivector field $\Lambda \in V \wedge V \backslash \{0\}$ if and only if V admits a one-dimensional trivial Lie algebra representation within $V \wedge V$.

Examples:

a) Lie algebra $\mathrm{I}_{16} = \langle \partial_x, \partial_y, x\partial_x - y\partial_y, x\partial_y, \ldots, x^r\partial_y \rangle$. We have

$$[\partial_x, \partial_x \wedge \partial_y]_{\mathrm{SN}} = 0, \qquad [x\partial_x - y\partial_y, \partial_x \wedge \partial_y]_{\mathrm{SN}} = -\partial_x \wedge \partial_y + \partial_x \wedge \partial_y = 0,$$
$$[\partial_y, \partial_x \wedge \partial_y]_{\mathrm{SN}} = 0, \qquad [x^j\partial_y, \partial_x \wedge \partial_y]_{\mathrm{SN}} = -jx^{j-1}\partial_y \wedge \partial_y = 0, \qquad j = 1, \ldots, r.$$

Hence, $W = \langle \partial_x \wedge \partial_y \rangle \subset V \wedge V$ is a trivial one-dimensional representation of V. As a consequence, $\partial_x \wedge \partial_y$ turns all the elements of W into Hamiltonian vector fields.
b) Lie algebra $\mathrm{I}_{14B} = \langle \partial_x, \partial_y, \eta_2(x)\partial_y, \ldots, \eta_r(x)\partial_y \rangle$. The Lie derivatives of $\Lambda := \partial_x \wedge \partial_y \in V \wedge V$ with respect to elements of I_{14B} vanish:

$$[\partial_x, \partial_x \wedge \partial_y]_{\mathrm{SN}} = [\partial_y, \partial_x \wedge \partial_y]_{\mathrm{SN}} = 0, \qquad [\eta_j(x)\partial_y, \partial_x \wedge \partial_y]_{\mathrm{SN}} = -\frac{\partial \eta_j}{\partial x}\partial_y \wedge \partial_y = 0,$$

for $j = 2, \ldots, r$. This turns $W = \langle \partial_x \wedge \partial_y \rangle$ into a trivial one-dimensional representation of V. So, V consists of Hamiltonian vector fields relative to Λ.

Theorem 1

If V is a planar Vessiot–Guldberg Lie algebra admitting a two-dimensional ideal I such that $I \wedge I \neq \{0\}$ and the elements of V act on I by traceless operators, namely the mappings $\vartheta_X : Y \in I \mapsto [X, Y] \in I$ are traceless for each $X \in V$, then V becomes a Lie algebra of Hamiltonian vector fields with respect to every element of $I \wedge I \backslash \{0\}$.

Example: Lie algebra $\mathrm{P}_1 = \langle \partial_x, \partial_y, y\partial_x - x\partial_y \rangle$

admits a two-dimensional ideal $I = \langle \partial_x, \partial_y \rangle$ satisfying that $I \wedge I \neq \{0\}$. Moreover, $\partial_x, \partial_y, y\partial_x - x\partial_y$ act as traceless operators on I. In view of Theorem 1, the basis $\partial_x \wedge \partial_y$ of $I \wedge I$ becomes a Poisson bivector turning P_1 into a Lie algebra of Hamiltonian vector fields.

Example: Lie algebra $\mathrm{P}_5 = \langle \partial_x, \partial_y, y\partial_x, x\partial_y, x\partial_x - y\partial_y \rangle$

We have that $I = \langle \partial_x, \partial_y \rangle$ is an ideal of P_5 with $I \wedge I \neq 0$ and it is straightforward to prove that all elements of P_5 act as traceless linear operators on I. Hence, Theorem 1 ensures that P_5 is a Lie algebra of Hamiltonian vector fields relative to the basis $\Lambda := \partial_x \wedge \partial_y$ of $I \wedge I$.

The above method works for most non-simple Lie algebras of Hamiltonian vector fields

Classification of Lie algebras of Hamiltonian vector fields on the plane

#	Primitive	Hamiltonian functions h_i	ω	Lie–Hamilton algebra
P_1	$A_0 \simeq \mathfrak{iso}(2)$	$y,\ -x,\ \frac{1}{2}(x^2+y^2),\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{iso}}(2)$
P_2	$\mathfrak{sl}(2)$	$-\dfrac{1}{y},\ -\dfrac{x}{y},\ -\dfrac{x^2+y^2}{y}$	$\dfrac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
P_3	$\mathfrak{so}(3)$	$\dfrac{-1}{2(1+x^2+y^2)},\ \dfrac{y}{1+x^2+y^2},\ -\dfrac{x}{1+x^2+y^2},\ 1$	$\dfrac{\mathrm{d}x \wedge \mathrm{d}y}{(1+x^2+y^2)^2}$	$\mathfrak{so}(3)$ or $\mathfrak{so}(3) \oplus \mathbb{R}$
P_5	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$y,\ -x,\ xy,\ \frac{1}{2}y^2,\ -\frac{1}{2}x^2,\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{sl}(2) \ltimes \mathbb{R}^2} \simeq \mathfrak{h}_6$
#	Imprimitive	Hamiltonian functions h_i	ω	Lie–Hamilton algebra
I_1	$\mathbb{R}$	$\int^y f(y')\mathrm{d}y'$	$f(y)\mathrm{d}x \wedge \mathrm{d}y$	$\mathbb{R}$ or $\mathbb{R}^2$
I_4	$\mathfrak{sl}(2)$ (type II)	$\dfrac{1}{x-y},\ \dfrac{x+y}{2(x-y)},\ \dfrac{xy}{x-y}$	$\dfrac{\mathrm{d}x \wedge \mathrm{d}y}{(x-y)^2}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
I_5	$\mathfrak{sl}(2)$ (type III)	$-\dfrac{1}{2y^2},\ -\dfrac{x}{y^2},\ -\dfrac{x^2}{2y^2}$	$\dfrac{\mathrm{d}x \wedge \mathrm{d}y}{y^3}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
I_8	$B_{-1} \simeq \mathfrak{iso}(1,1)$	$y,\ -x,\ xy,\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{iso}}(1,1) \simeq \mathfrak{h}_4$
I_{12}	$\mathbb{R}^{r+1}$	$-\int^x f(x')\mathrm{d}x',\ -\int^x f(x')\xi_j(x')\mathrm{d}x'$	$f(x)\mathrm{d}x \wedge \mathrm{d}y$	$\mathbb{R}^{r+1}$ or $\mathbb{R}^{r+2}$
I_{14A}	$\mathbb{R} \ltimes \mathbb{R}^r$ (type I)	$y,\ -\int^x \eta_j(x')\mathrm{d}x',\quad 1 \notin \langle \eta_j \rangle$	$\mathrm{d}x \wedge \mathrm{d}y$	$\mathbb{R} \ltimes \mathbb{R}^r$ or $(\mathbb{R} \ltimes \mathbb{R}^r) \oplus \mathbb{R}$
I_{14B}	$\mathbb{R} \ltimes \mathbb{R}^r$ (type II)	$y,\ -x,\ -\int^x \eta_j(x')\mathrm{d}x',\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{(\mathbb{R} \ltimes \mathbb{R}^r)}$
I_{16}	$C^r_{-1} \simeq \mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$	$y,\ -x,\ xy,\ -\dfrac{x^{j+1}}{j+1},\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}}$

Examples of Lie–Hamilton systems on the plane

Table: Specific Lie–Hamilton systems according to the GKO classification.

LH algebra	#	LH systems
$\mathfrak{sl}(2)$	P_2	Milne–Pinney and Kummer–Schwarz equations with $c > 0$
		Complex Riccati equation
$\mathfrak{sl}(2)$	I_4	Milne–Pinney and Kummer–Schwarz equations with $c < 0$
		Split-complex Riccati equation
		Coupled Riccati equations
		Planar diffusion Riccati system for $c_0 = 1$
$\mathfrak{so}(3)$	P_3	Integrable system with trygonometric non-linearities
$\mathfrak{sl}(2)$	I_5	Milne–Pinney and Kummer–Schwarz equations with $c = 0$
		Dual-Study Riccati equation
		Harmonic oscillator
		Planar diffusion Riccati system for $c_0 = 0$
$\mathfrak{h}_6 \simeq \overline{\mathfrak{sl}(2) \ltimes \mathbb{R}^2}$	P_5	Dissipative harmonic oscillator
		Second-order Riccati equation in Hamiltonian form
$\mathfrak{h}_2 \simeq \mathbb{R} \ltimes \mathbb{R}$	$I_{14A}^{r=1}$	Complex Bernoulli equation
		Generalised Buchdahl equations
		Lotka–Volterra systems

Outlook

- We are studying the classification of analytic and smooth finite-dimensional Lie algebras of vector fields on the plane around singular points and the general three-dimensional one.
- We are studying the classification of finite-dimensional Lie algebras of smooth vector fields on the real line around singular points.
- Applications to the stability of systems are being analysed.